Estel Bradley Murdock
162 S. 500 E. #204
Provo, Utah 84606

Nature's Way of Measuring

by Estel B. Murdock

(inspired by Buckminister Fuller)

Chapter One

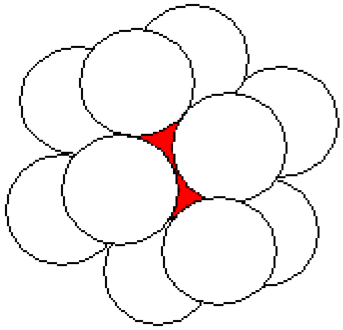
The Isotropic Vector Matrix

Measurement depends upon your frame of reference. Being a scientist, I am very optimistic, that is, I want to optimize on every measurement, to use the simplest possible means. For example, the shortest distance between two points is a straight line. If your plane is curved, the shortest distance requires you to go outside the plane. A tunnel through the Earth's crust is straighter than a path along the surface (if geology is not considered).

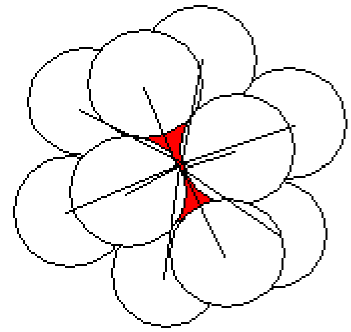
Traditional mathematics is based upon the unit square as the optimum measure and calls it the simplest form. It requires each side to be of unit measure. Therefore, the diagonal of a square becomes $\sqrt{2}$. But in reality, the triangle is the simplest form.

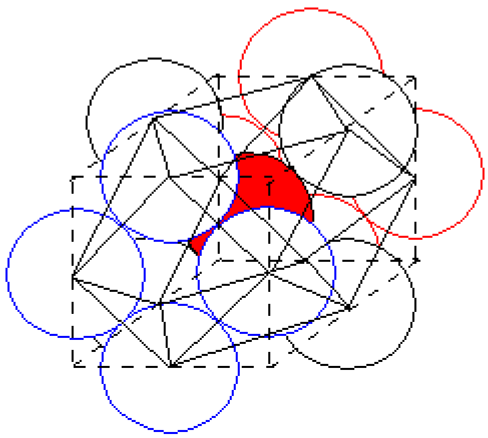
If the diagonal of the square is chosen to be the criteria for unity instead of the side, then we admit the triangle to be the simplest form and the side of the square becomes $1/\sqrt{2}$ (which is equal to $(\sqrt{2})/2$ for people who are uncomfortable with an irrational in the denominator.)

The main question I want to answer in this book is how many equilateral triangles or tetrahedrons instead of how many squares or cubes. Why? I'll answer that question below. First, let me introduce you to Buckminister Fuller's idea of the Isotropic Vector Matrix.



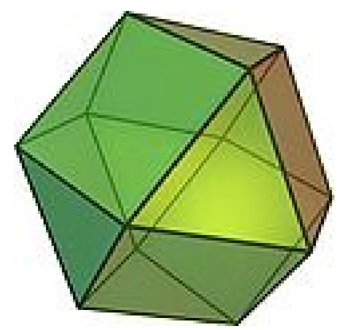
The **Isotropic Vector Matrix** comes from the closest packing of unit radius spheres. Each sphere within the isotropic vector matrix has 12 surrounding spheres. Connecting the centers of each of the 12 spheres to the center of the nuclear sphere are 12 double radii radiating from the nuclear sphere. (One radius from each sphere connected to one radius from the nuclear sphere.) Each axis is separated by 60^o from an adjacent axis. This angle of 60^o is a property of the adjacency of the spheres.



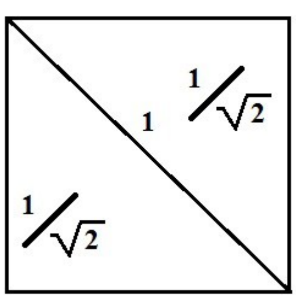


Stretched open to show the inside sphere.

These radii form four hexagonal planes. Connecting the ends of the radii together with double radii connects each sphere into a matrix. Since the radius of a sphere is called a vector, this matrix is called an Isotropic Vector Matrix. With all thirteen spheres connected at their centers, we form a cuboctahedron, which is a cube with its corners sliced off. It is intersected by the four hexagons. Buckminister Fuller called this the **Vector Equilibrium**.



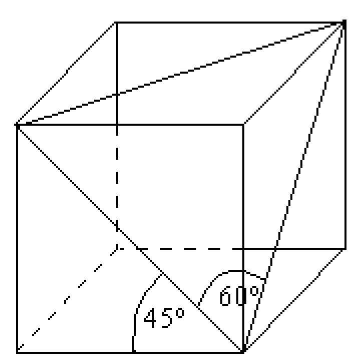
The Natural Way to Measure



If the diagonal is designated unity from the beginning, then the Pythagorean Theorem says the sides become equal to $1/\sqrt{2}$ or $\sqrt{2}/2$. This is the cos 45^o or the sin 45^o .

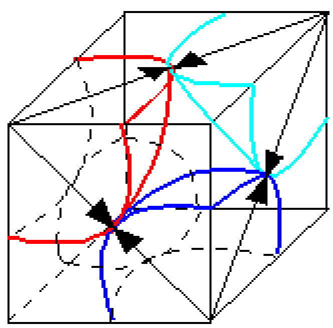
Now, if we use this diagonal of 1 unit as a radius of a circle, generalizing the square to become a rectangle, we get all the angles between 0^o and 90^o. Therefore, we get all the values of sin θ and cos θ. The diagonal is a whole number separated from irrational numbers by 45^o .

Taking this to a higher dimension is a cube showing diagonals on each side producing 5 tetrahedrons on the inside. The diagonals are 60^o away from each other on the inside because three side diagonals make an equilateral, equiangular triangular plane with inside angles of



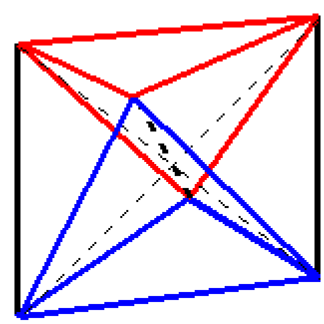
60°, but these diagonals are 45° away from the edges of the cube.

The 60° triangle is at a 45° angle to the edge of the cube, therefore, the **isotropic vector matrix** coincides with, but is separated from the **cubic vector matrix** by 45°. Only the **diagonals** of the cube's faces are set within the isotropic vector matrix!

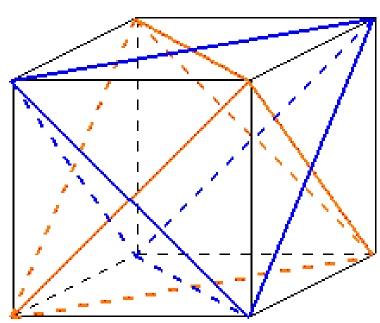


Four $^1/_8$ unit spheres within the isotropic vector matrix come together within the cubic vector matrix such that four of the corners correspond to the centers of the spheres, and the cube's face diagonals coincide with the sphere's radii, making the diagonals of the square sides two units each, that is, two opposing vectors. Therefore, each square side on the cube has a diagonal of 2, and each edge of the cube has a length of $\sqrt{2}$. The area of each one of these squares is $(\sqrt{2})^2 = 2$ traditional unit squares. But for our purposes, a unit square has a unit diagonal with sides of $\sqrt{2}$.

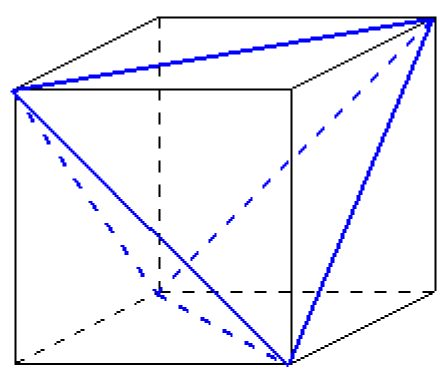
The traditional area of the cube is $(\sqrt{2})^3 = 2.828428$.



The unit octahedron is made up of four unit tetrahedrons. Therefore, the volume of the octahedron is 4 tetrahedrons. Cut two of these tetrahedrons in half to make four ½ – tetrahedrons, each having a volume of $^1/_2$. When four ½ – tetrahedrons are added to each face of the unit tetrahedron, the smallest cube is created because $4(^1/_2) + 1 = 3$, an easier way of calculating the smallest cube than using the $\sqrt{2}$.



The diagonals of the six faces of a cube which only line up with the isotropic vector matrix form two interlacing tetrahedrons. The tetrahedron is therefore more fundamental than a cube for measuring volume. Taking one of these tetrahedrons and adding the right kind of tetrahedron to each one of the four faces, it takes five tetrahedrons to make one cube.



Comparing this rational volume of 3 to the calculated volume using the $\sqrt{2}$, we get the synergetics conversion factor of $^3/_{2.828428} = 1.06066$. Using this conversion factor on conventional areas and volumes, they are converted to rational areas and volumes. This conversion factor of $1.06066 = \sqrt{(9/8)}$.

For areas, 2 dimensions, $\sqrt{(9/8)}$ is triangled to become $(\sqrt{(9/8)})^2 = {}^9/_8$.

For volumes, 3 dimensions, √(9/8) is tratrahedroned to become $(\sqrt{(9/8)})^3 = 1.193243$.

Here are some examples.

The Great Pyramid at Giza has a volume of 2.5 million cubic meters. 2.5 x 1.193243 = 3, that is 3 million tetrahedrons.

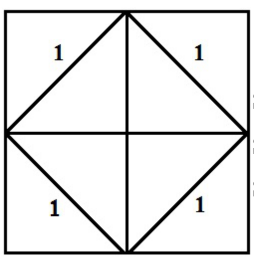
The Chalula Pyramid in Mexico has a volume of 4.45 million cubic meters. 4.45 x 1.193243 = 5 million tetrahedrons.

The numbers become rational.

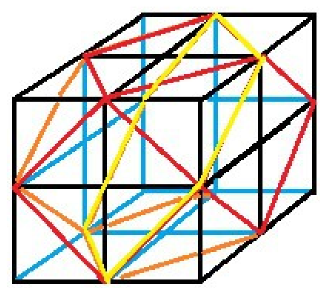
(Triangled and tetrahedroned will be explained later, being equivalent with squaring and cubing.)

The traditional way of calculating uses only squares and cubes, locking mathematicians into using irrational numbers to provide precise measurements. This is true because both the square and the cube are not units of measure. They are divisible, so they have to make up for this by using irrational numbers or parts of squares or cubes.

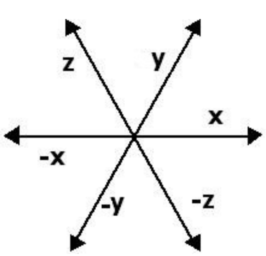
What we have here is that the square or the cube are not basic and fundamental shapes that form the basis of a coordinate systems. The basic shapes as we have seen are the triangle and the tetrahedron. General coordinate systems need to be interpreted on the basis of the primary and most basic of shapes, the triangle and tetrahedron which are indivisible.



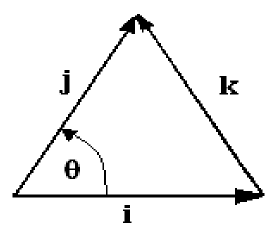
Doubling the original square, we have 4 smaller squares whose diagonals produce an inner square with unit sides separated 45° from the parent square.



Taking this into 3 dimensions and following the unit diagonals around the cube, we find inside the cube 4 hexagons, each one of which is in a different coordinate system 45° away from what the cube exists in. In other words, within the 90° coordinate system, turned at an angle of 45° , is a 60° coordinate system.



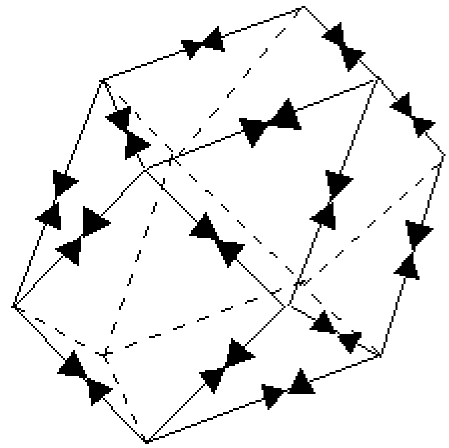
Each unit hexagon is divided into 6 unit triangles. The sides of these triangles are the axes of the hexagon. They correspond with the three x, y, z axes of the 90° coordinate system projected onto a plane. All three basis vectors can be brought together to form a triangle. This forms a coordinate system where

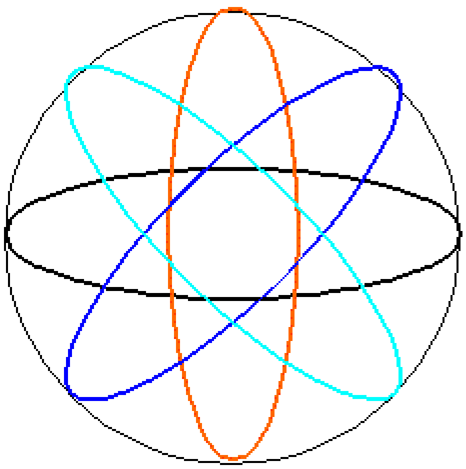


k = i + j. This negates the Pythagorean Theorem. Generalizing, z = x + y.

Notes:

Each edge of this cuboctahedron is formed from the double vector radii of two touching spheres, each vector opposing the other one. Adding up all the vectors results in a zero vector. Therefore, this cuboctahedron is named the **Vector Equilibrium**.





This relationship of the 12 points of contact around a sphere and the 12 centers of the 12 surrounding spheres and the 12 points of contact between the Vector Equilibrium and its encasing square are all brought together by rotating each of the hexagonal planes slicing through the Vector Equilibrium getting the four great circles of a sphere. This uses the four major axes of rotation of the Vector Equilibrium (through the 8 corners of the cube). Each circle touches 3 other circles. Each circle has three points of contact. That is 3 x 4 = 12. That cements the relationships. (Also, the 3 dimensions of the sphere times the 4 great circles of the sphere equals 12.)

Chapter Two

Volume

Volume has traditionally been measured using the cube. N^3 (N cubed) is a cubic volume and is three edges of a cube multiplied together. (Each edge being divided by an equal number.) If we measure volume using the tetrahedron as the basic unit of measure, N^3 (called N tetrahedroned) is three edges of the tetrahedron multiplied together. Dividing each side of the cube into n squares, the cube is divided into n^3 cubes. Dividing the tetrahedron into n^3 does not work, for the tetrahedron and the cube do not have a one–to–one correspondence.

n	For a cube, n^3	For a tetrahedron, n^3
1	1	1
2	8	7
3	27	26
4	64	52
5	125	107
etc.	etc.	etc.

The volume of a cube and the volume of a tetrahedron is different. The cube is an all space filler, but the tetrahedron is not. But combined with the octahedron having a volume of 4, the combination of octahedron and tetrahedron with a volume of 1 fills all of space. So counting the octahedrons in each succeeding layer of the tetrahedron where each layer is of the same height, we get the triangular numbers of octahedrons as 0, 1, 3, 6, 10, 15, 21, 28, etc. The count of the volumes of octahedrons in each layer added to the count of the volumes of tetrahedrons in each succeeding layer give the volume of each layer of a tetrahedron:

tetrahedral volumes	+	octahedral volumes	=	Volume of Tetrahedron layer	
1		0		1	(equal to n
3		4		7	tetrahedral
7		12		19	volumes)
13		24		37	
21		40		61	
etc.		etc.		etc.	

so that the volume of each succeeding tetrahedron is the first plus the second plus the third, etc.

$1 + 7 = 8 \quad = 2^3$

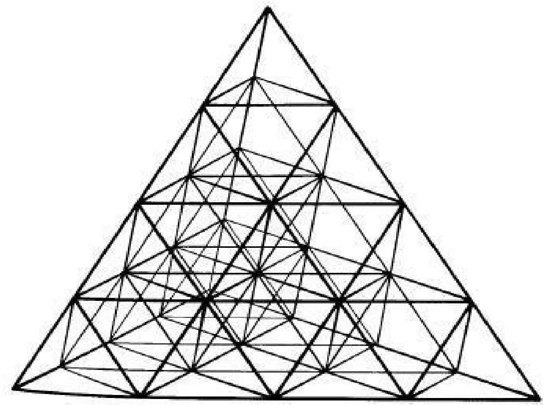
$8 + 19 = 27 \quad = 3^3$

$27 + 37 = 64 \quad = 4^3$

$64 + 61 = 125 \quad = 5^3$

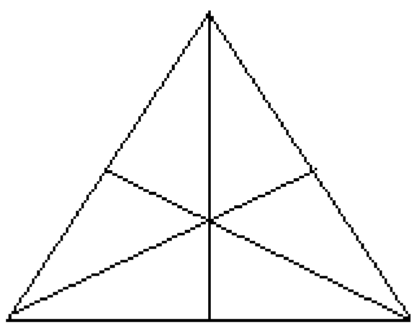
etc. etc.

Notice that the list of volumes now correspond with the volumes of cubes, so the addition of octahedrons to tetrahedrons gives a one–to–one correspondence of cubes and tetrahedrons.

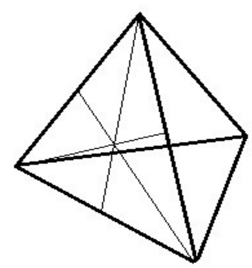


Therefore, a number n tetrahedroned, that is, n^3, corresponds to a number cubed.

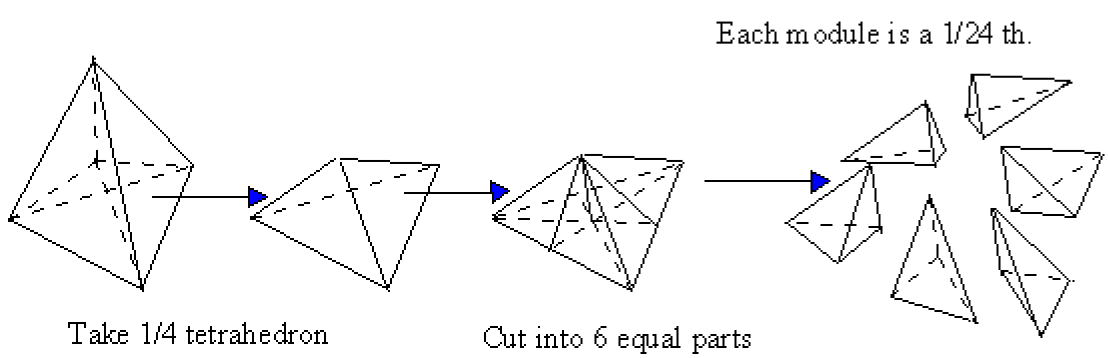
Volume as the Tetrahedral Part



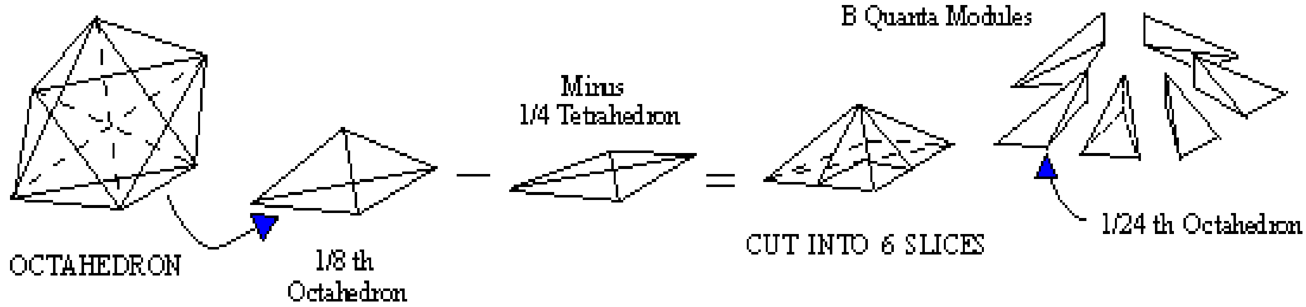
The equi–angled, equi–edged triangle has three axes from each corner to opposite side dividing the triangle into 6 right triangles.

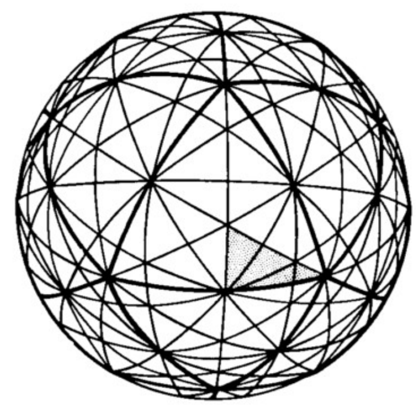


In a unit tetrahedron, these axes on each of the faces of the tetrahedron become planes within the tetrahedron, and they divide the tetrahedron into 4 x 6 = 24 modules, each one being called the A–Quanta Module.



A whole octahedron has a volume of 4. $1/8^{th}$ of that is a volume of **½. T**aking away a quarter of a tetrahedron, which is **¼** volume, gives you a quarter volume. Dividing that by 6 gives you a $1/24^{th}$ volume. Call that the B Quanta Module. The A and B Quanta Modules are equal in volume.





Neither the tetrahedron nor the octahedron are all space fillers. It takes both to fill all of space. That is why to describe any part of space, you need a collection of A and B quanta modules, A's coming from tetrahedrons, and B's coming from octahedrons.

Taking each of the 20 equilateral triangles of the spherical icosahedron (an icosahedron drawn on the surface of a sphere) and dividing them into 6 right triangles, you get 20 x 6 = 120 LCD (lowest common divisor) triangles. Any further subdivision is no longer similar, thus the LCD 60 positive, 60 negative triangles.

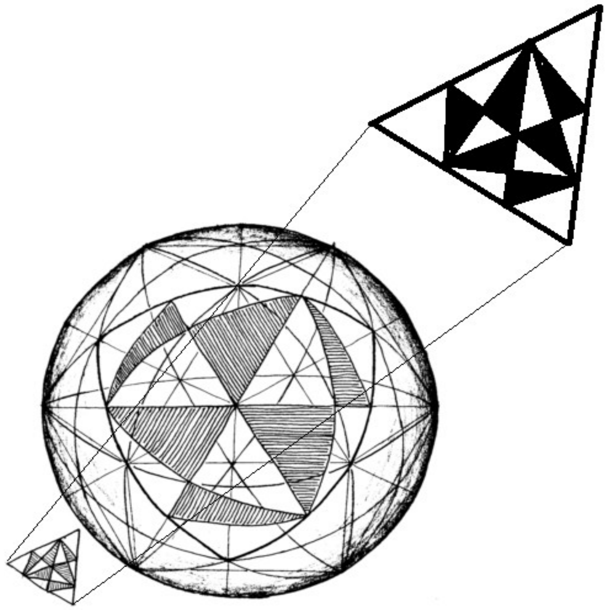
Taking one of the A–Quanta Modules and unfolding it, you get an LCD triangle. The LCD triangle is $^1/_{120}$ th of surface area division caused by 15 great circles of the spherical icosahedron.

Volume of a Sphere

For a unit sphere, the radius being one, the conventional volume is $4/3\ \pi\ r^3 = 4.188790$ cubes. Multiplying by the synergetics constant for 3 dimensions, $[\sqrt{(9/8)}]^3$, that is, 4.188790 x 1.193243 = 4.998425. That is 5 tetrahedrons. It has therefore become rational and the new formula for spherical volume is $V = 5r^3$. An easier way of getting the volume of the unit sphere is dividing the surface into 120 LCD triangles and extending them to the center to form 120 tetrahedrons, each having the volume of an A or B Quanta Module, each one having a volume of $1/24^{th}$ of a unit tetrahedron. Therefore,
$120 \times {}^1/_{24} = 5$, the volume of the unit sphere.

The volume of the unit cube is 3. Taking away the 8 corners, such that each is $^1/_{16}$ th of a unit tetrahedron, produces the Vector Equilibrium. $8 \times {}^1/_{16} = ½$, so the sum of the corners taken away is ½ of a unit tetrahedron, showing that the volume of a unit or basic Vector Equilibrium is 3 – ½ = 2 ½ . Therefore, the volume of a unit sphere, being 5, is the same as the volume of two Vector Equilibriums and has the same volume as 120 A and B Quanta Modules. (The volume of the rhombic dodecahedron can be found similarly using A and B Quanta Modules.)

All symmetric forms can be measured simply using the tetrahedron as the unit of measure. This is without the use of π.



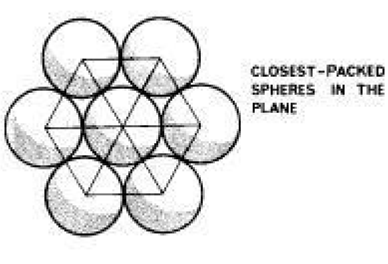
There is a ratio involved in the volume of a sphere. It is 15/3. It comes from (120/8)/(24/8), 8 because of the spherical octahedron where there are 8 faces. The spherical right triangle within the spherical equilateral triangle is $1/120^{th}$ of the surface area. In both the spherical icosahedron and the spherical octahedron, there are 15 A and B quanta modules in one of the spherical triangular faces.

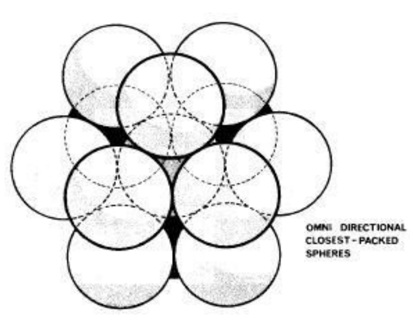
Another ratio is 20/4 related to the spherical cuboctahedron made up of 60 A and B Quanta Modules. This also has to do with the volume of a sphere. (15/3) x 4 = 60/12 = (20/4) x 3. There are 4 planes in the Vector Equilibrium and three axes in each of the 4. The Vector Equilibrium is the key to the reason why the unit sphere is 5. The cuboctahedron has 60 A and B Quanta Modules. 60/12 = 5.

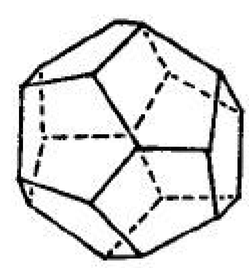
There is a relationship between the icosahedron, the cuboctahedron, and the dodecahedron. The cuboctahedron and the icosahedron have the same number of vertexes, where the closest packing of spheres have their centers. Taking out the central sphere from the cuboctahedron, it contracts to a more symmetrical configuration, the icosahedron, but the number of vertexes, which is 12, remain constant. The dodecahedron has 20 vertexes, but it has 12 faces. The spherical dodecahedron thus has 12 equally spaced centers on its surface where the vertexes of the encased icosahedron touches the surface, as does the cuboctahedron. So the relationship of each of these spheres is this 12 equally spaced points. This obviously comes from the way 12 spheres pack closely around a central sphere. Thus common denominator of 12.

Formation of the Dodecahedron

Take 6 circles surrounding one circle and push them onto the center circle with equal force**.** The center circle becomes a hexagon. Beehives are the result of the most economical use of circular space. The dodecahedron represents the most economical use of three dimensional space, and like the cube is an all–space filler. If you take 12 soft spheres surrounding a central soft sphere and each outer sphere is pushed with the same force towards the central sphere, a dodecahedron results.







Dodeca-hedron

Sample Volumes

This chart shows some examples of the volumes of some solids based on the volume of the tetrahedron as unity.

SYMMETRICAL FORM (based upon the closest packing of unit radius spheres)	***TETRA VOLUMES*** (the unit of volume being one unit tetrahedron)	***A and B QUANTA MODULES*** (multiples of 12 spheres surrounding nuclear sphere)
Tetrahedron	1	24 = 2 x 12
Vector Equilibrium	2 ½	60 = 5 x 12
Cube	3	72 = 6 x 12
Octahedron	4	96 = 8 x 12
Nuclear Sphere	5	120 = 10 x 12
Rhombic Dodecahedron	6	144 = 12 x 12

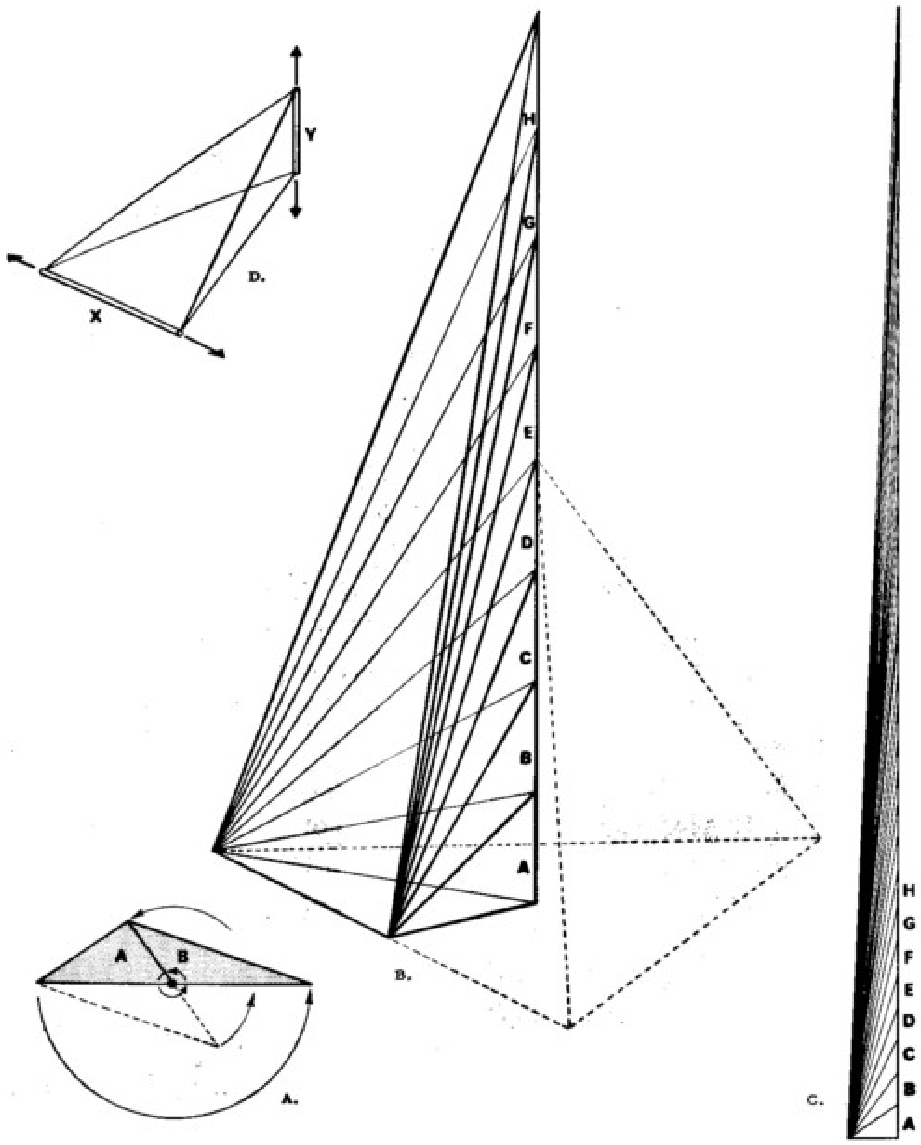
All of the above symmetrical forms are of the form $\mathbf{Nr^3}$ where N is the number of tetrahedrons in the volume of the basic form and r is the frequency of that form. For example, using r as equal to 2nd frequency, $r^3 = 2^3 = 8$: (frequency as expansion of time through space)

SYMMETRICAL FORM Nr^3	***TETRA VOLUMES*** r^3 (r = 2)	***A and B QUANTA MODULES***
tetrahedron	8	08 x 24 = 192 = 8 x 24
vector equilibrium	20	20 x 24 = 480 = 8 x 60
cube	24	24 x 24 = 576 = 8 x 72
octahedron	32	32 x 24 = 768 = 8 x 96
nuclear sphere	40	40 x 24 = 960 = 8 x 120
rhombic dodecahedron	48	48 x 24 = 1152 = 8 x 144

Remember that an A or B quanta module is $1/24^{th}$ volume.

The number of A and B quanta modules are shown here as multiples of tetrahedrons of frequency 2 times the number of A and B quanta modules in the primary forms.

The number of A and B quanta modules are also the same number as important angles with a system of angles, lines and planes making up three dimensional forms.



(from Buckminister Fuller's Synergetics)

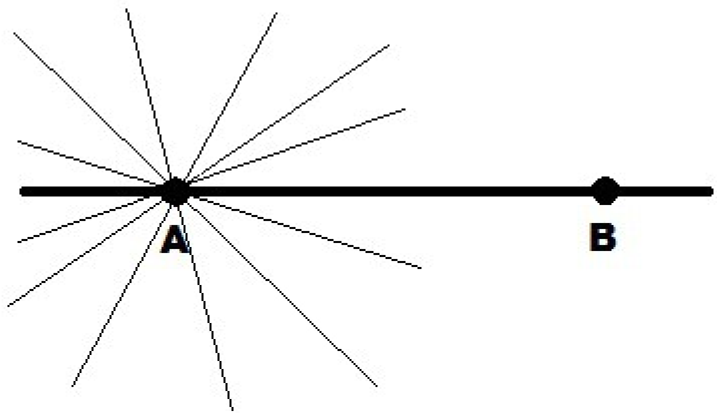
A comparison of the end views of the A and B Quanta Modules shows that they have equal volumes by virtue of the fact that they have equal base areas and identical altitudes.

It follows from this that if a line, originating at the center of area of triangular base of the regular tetrahedron, is projected through the apex of the tetrahedron to infinity, is subdivided into equal increments, it will give rise to additional Modules to infinity. Each additional Module will have the same volume as the original A or B Module, and as the incremental line approaches infinity the Modules will tend to become lines, but lines still having the same volume as the original A or B Module.

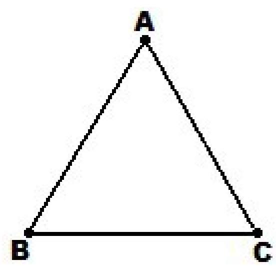
End view shows Modules beyond the H Module shown in (B).

The two discrete members X and Y can move anywhere along their respective axes and the volume of the irregular tetrahedron remains constant. The other four edges vary as required.

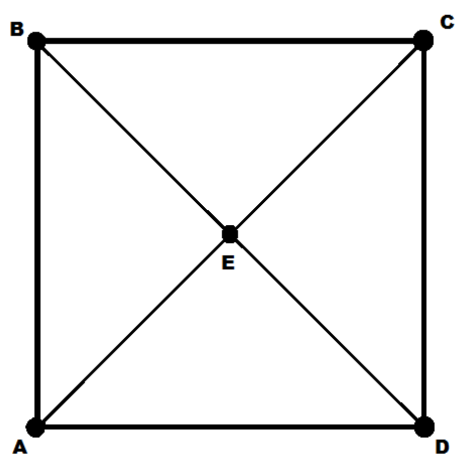
Chapter Three
Area



You can draw an infinite number of lines through a single point, but you can only draw a single line through two points.

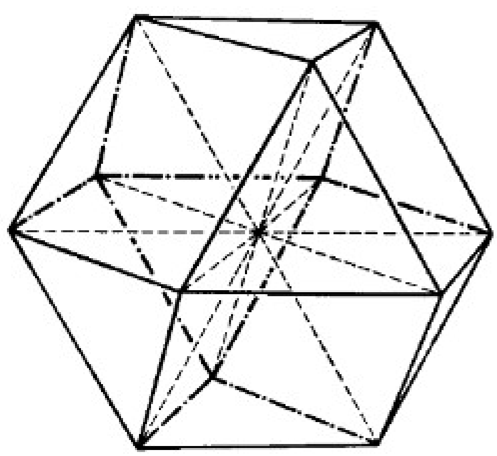


Drawing one line through any two points, you can only draw three lines through three points.

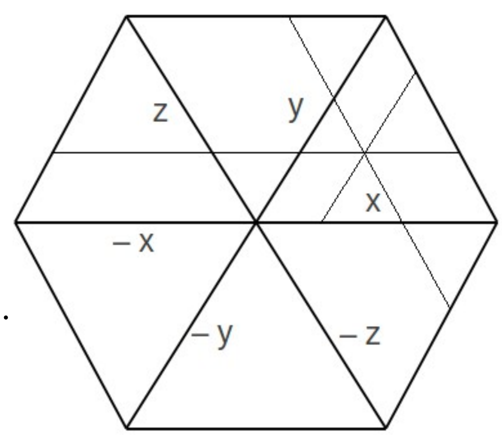


But if you try to draw all the lines you can through a group of four points using the postulate of drawing one line through any two points, you cannot draw any less than six lines, and therefore, drawing a simple square is out of the question. Drawing one line through two points at a time among a group of four points, gives you two diagonals as well the sides of a square. Then there is an additional fifth point. That is because the postulate stated says that you have to include the lines between the corner points. Therefore, you cannot simply draw four lines through four points. You wind up with four triangles, the simplest planar figure, instead of a simple square.

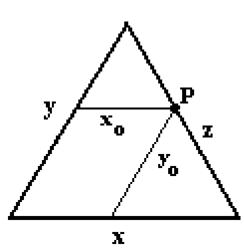
The Hexagonal Plane



The Vector Equilibrium is a 4–dimensional manifold. It exists 45° from an enclosing cube. It can be sliced four ways to produce 4 different planes and has 4 axes, each axis being coplanar with each of the 4 planes. Each one of these planes is a hexagon. The hexagon has three axes, each one drawn from corner to opposite corner, all three meeting in the center and forming six equiangular equilateral triangles. Each one of these hexagons in the Vector Equilibrium is a 3–dimensional manifold projected onto a plane for 2–dimensional measurement.



The hexagon represents a cube with its x, y, and z axes. Each point within the hexagon is P(x, y, z) and is found by taking a 60° orthogonal leg from each of the three axes. First, there are three areas to deal with: (x, y, z), (– x, – y, z), and (x, – y, – z). These can be divided into sextants: (x, y, z), (y, z, x), (z, – x, – y), (– x, – y, z), (– y, – z, x), and (– z, x, – y). It must be remembered that each axis extends away from the central point called the origin of the hexagon. For example, a line z_l parallel to the z axis passing through the origin extends from the origin to the end of the y and x axes to form an equilateral triangle. This line z_l is still considered to be the z axis. In like manner, each point inside the hexagon has an x, y and z axis passing through it.



P(x, y, z) in the (x, y, z) sextant is shown on the z side of an equilateral triangle with an x axis extending from the y side, and a y axis extending from the x side. So for each sextant, only two coordinates and one side of an equilateral triangle is needed to determine a point.

A hexagon represents one cycle of six. One sixth of the hexagon is the 60° triangle we want to deal with in this chapter.

Each and Every Triangle Has a One–To–One Correspondence with a 60° Triangle

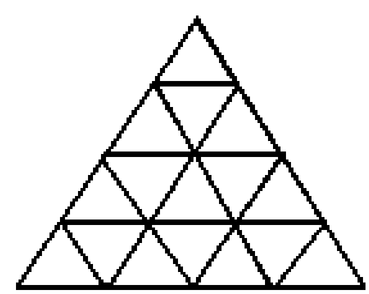
Divide the sides of an equiangular, equilateral triangle into n equal line segments. Take another arbitrary triangle covering the same area as the first triangle with arbitrary sides and internal angles. Each side of the second triangle can be divided into n equal line segments. The length of the line segments on one side of the triangle does not have to match the length of the line segments on any other side of the triangle as long as they are the same number of line segments. Then each side of the second triangle has a one–to–one correspondence with a corresponding side of the first triangle. Therefore, any triangle can be represented by an equiangular, equilateral triangle having the same area. Dividing each triangle into smaller triangles by connecting the segment ends of one side to the segment ends of the opposite side using parallel lines (each side having two opposite sides), then the number of smaller triangles in one triangle is equal to the number of smaller triangles in the other triangle. Any arbitrary triangle is therefore but a distortion of an equiangular, equilateral triangle. This may be advantageous by simplifying the solving of triangular problems. One such problem is the representation of the triangling of a number by any triangle.

Triangling a Number

Triangles are more basic for measurement than are squares because the triangle is the most simple of the polygons. Any number n squared $n^2 = n \times n$. In Nature's Way of Measuring, it is called n triangled, because it is the multiplication of two sides of an equilateral triangle instead of the two sides of a square. Dividing all sides of a square by n and connecting each point to its opposite point with a line, the square is divided into n^2 (squared) squares. If the sides of an equilateral triangle are divided by n, and each point is connected to its two opposite points (at 60^o angles) with a line, the triangle is divided into n^2 (triangled) triangles. The number n has been triangled.

The Area of a Triangle

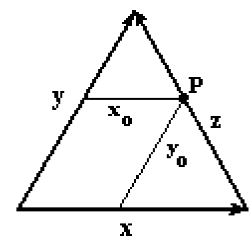
The square of a number n, n^2, has a one–to–one correspondence with the triangle of the number n, n^2. Dividing a square into n^2 similar squares, is the same number when you divide an equilateral triangle into n^2 similar triangles. In the figure, $4^2 = 16$ triangles. The triangle of a number is the area of an equiangular equilateral triangle. This can be generalized into any triangle. Also, if nm is the area of any rectangle, then ½ nm is the area of any triangle where n is the base and m is the height of the triangle. But if we substitute the irregular triangle with its equivalent area equilateral triangle, taking n as the divisor of any side, then $\frac{1}{2} nm \approx n^2$. ½ nm is in squares, but n^2 is in triangles. So the area of any triangle can be expressed as n^2.



Finding the Triangular Root of a Number

Let an equilateral triangle be divided into n^2 similar triangles. By the definition of triangling a number n, each side of the triangle is divided into n parts. Therefore, the triangular root $\sqrt{(n^2)} = n$. Now if $t = n^2$, then $\sqrt{t} = n$. The triangular root of t is equal to n, where t is the number of similar triangles within an equilateral triangle with sides measuring n units. The triangular root becomes the scale of any triangle.

The Equation of a Triangle

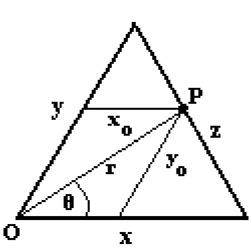


A vector is a directed line segment and will be represented as a bold faced letter, for example **a**, **b**, **c** or **x**, **y**, **z**. The addition of two vectors **x** and **y** is **x** + **y** = **z**. This is also the equation of a triangle. This equation can be used to find any point on the line segment |**z**|. In an equiangular, equilateral triangle, draw a line parallel to |**x**| from |**y**| to a point P on |**z**|. That gives the coordinate x. Draw a line parallel to |**y**| from |**x**| to the point P on |**z**|. That gives the coordinate y. x + y = z, where z is the length |**z**| . The reason this is so is that both x and y are sides of smaller equiangular, equilateral triangles within the larger triangle with one side congruent to and as part of **z**, the side of the larger triangle. The line segment on **z** above P is congruent to x, and the line segment on **z** below P is congruent to y. These two line segments can therefore be called x and y and add up to z, or the length |**z**|. (This will work with any triangle and is called generalizing the coordinates.)

Is Z an Imaginary Number?

In order to talk about 2–dimensional measurement, we must first talk about imaginary numbers. In Nature's way of measuring there is no imaginary numbers. Traditionally, $\sqrt{(-1)}$ is given the symbol i, and an imaginary number z = a + ib. (Some authors use j.) But the number a + ib can be represented as the number a + b or as the ordered pair (a, b) or (x, y) which is so similar to z = x + y that all references to imaginary numbers a + ib will be referred to from now on as a + b or x + y, both of which is the imaginary number z or the coordinate z or a point (x, y) on the line z. If x, y, and z are unit vectors, then cz = ax + by or $c|_z = a|_x + b|_y$ (the z, x, y parts of a number) are also replacements of the imaginary number z = a + ib.

The logarithmic representation of a + ib is $re^{i\theta}$. Since we are representing a + ib as a + b, then $re^{i\theta}$ can be replaced with re^{θ}. When $\theta = \omega t$, then $a + b = re^{\omega t}$ and is a vector rotating in a counterclockwise rotation with an angular velocity of ω . For addition, a + b is used, and for multiplication, re^{θ} is used so the exponents only need to be added.



The complex number a + b can also be written as (cos θ + sin θ) where a = cos θ and b = sin θ . Also, e^{θ} = (cos θ + sin θ) and therefore, re^{θ} = r(cos θ + sin θ). The conjugate of z is – y – x, and the conjugate of – z is y + x. A complex number is defined as the endpoint of any vector, and a complex plane, any plane in which a vector is drawn from the origin out to the z–axis. If you plot a complex number or a vector in the complex plane (in other words, a plane in which a vector is drawn), then r will be the distance from the origin to the point on the z axis and θ will be the angle the vector makes with the x–axis.

DeMoivre's Formula[1]

DeMoivre's formula is the following:

$$(\mathrm{Cos}(\theta) + i\mathrm{Sin}(\theta))^n = \mathrm{Cos}(n\theta) + i\mathrm{Sin}(n\theta) \text{ where } 0^o \le \theta \le 90^o.$$

Using a variation of this formula, let us use

$$(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^n = \mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta) \text{ where } 0^o < \theta \le 60^o.$$

This formula is useful when you have a complex number and want to raise it to some power without doing a lot of work.

Write the complex number re^{θ} as $r\mathrm{Cos}(\theta)+r\mathrm{Sin}(\theta)$ and raise it to a power n.

Essentially what you are doing is taking a complex number of the form

1–Taken from Doctor Benway, The Math Forum at http://mathforum.org/dr.math/

$$a + b, \text{ and}$$

converting it to the form

$$re^{\theta},$$

raising it to a power in that form, then converting back to the first form. Observe:

$$\begin{aligned}(r\mathrm{Cos}(\theta) + r\mathrm{Sin}(\theta))^n &= (r(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^n \\ &= (r^n)(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^n \\ &= (r^n)(e^{\theta})^n \\ &= (r^n)(e^{n\theta}) \\ &= (r^n)(\mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta))\end{aligned}$$

Of course knowing DeMoivre's formula allows us to go straight from

$$(r(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^n$$

to

$$(r^n)(\mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta)).$$

Tetrahedral Roots of Numbers as Planes[1,2]

It will be found that the 3–dimensional manifold of a 60^o coordinate system can be obtained from a 90^o coordinate system using cubic roots and translated into tetrahedral roots of a system.

1–Taken from Doctor Anthony, The Math Forum at <http://mathforum.org/dr.math/>

2–The trigonometry here is based upon the hexagon and a 60^o cycle.

If $x^3 = N$, where N is some expression (which could be a constant), then you have a third degree equation, so there must be three roots.

Suppose $z^3 = 8$, z being a complex number. Now taking the tetrahedral root of each side (as if each edge of the tetrahedron with volume of 8 is divided into 2) you have z = 2, however, there are two other tetrahedral roots for this equation.

$$\text{Let } z^3 = 8(1 + 0).$$

$$(\text{Remember that } 0^o < \theta \leq 60^o)$$

But since Cos(6k) = 1 and Sin(6k) = 0 where k is any integer, we could write the equation as

$$z^3 = 8(\text{Cos}(6k) + \text{Sin}(6k)).$$

Take the tetrahedral root of both sides, and use DeMoivre's theorem which shows that:

$$z = [8(\text{Cos}(6k) + \text{Sin}(6k))]^{1/3}$$
$$z = [8^{1/3}(\text{Cos}(6/3k) + \text{Sin}(6/3k))]$$
$$z = 2[\text{Cos}(2k) + \text{Sin}(2k)]$$

Letting k = 0, 1, 2,

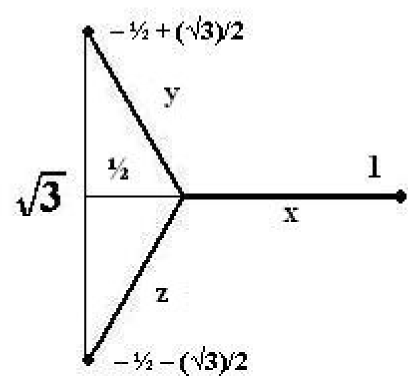
k=0 gives $z_1 = 2[(\text{Cos}(0) + \text{Sin}(0)]$
$= 2(1 + 0) = 2(1, 0)$
$= 2|_x$ (the one real root along the x–axis)
k=1 gives $z_2 = 2(\text{Cos}(2) + \text{Sin}(2))$ [2 is 1/3rd of a cycle of six.]
$= 2(-1 + 1) = 2(-1, 1)$
$= 2|_y$ (along the y–axis)
k=2 gives $z_3 = 2(\text{Cos}(4) + \text{Sin}(4))$ [4 is 2/3rd 's of a cycle of six.]
$= 2(-1 - 1) = 2(-1, -1)$
$= 2|_z$ (along the z–axis)

So, if $z^3 = 8$, we have the three roots of $2|_x$, $2|_y$ and $2|_z$, each 2 being on one of the three axes. If we give k more values, 3, 4, 5, … we simply repeat the three roots already found.

Let's do this over again, except this time, using imaginary numbers. First, we use one half of a unit equiangular, equilateral triangle. The height is (√3)/2, and the base is ½. Using the Pythagorean Theorem, the hypotenuse then is,

$$\sqrt{(((\sqrt{3})/2)^2 + (1/2)^2)} = 1.$$

So, if $z^3 = 1$, then $z^3 = (-\tfrac{1}{2} +/- ((\sqrt{3})/2))$



because half the vertical length between $1|_y$ and $1|_z$ is √3/2, and the horizontal length between that line and the imaginary axis is

½. Then, if $z^3 = 8$, we get

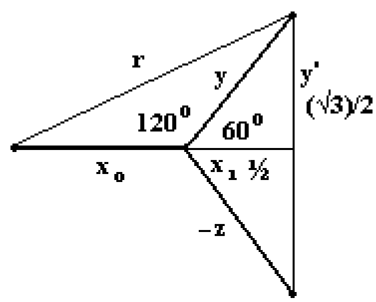
$$z^3 = -2 +/- 2\sqrt{3}\ .$$

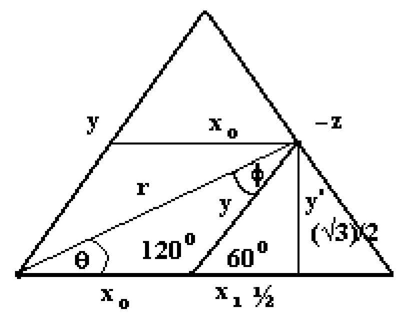
The $2|_y$ component is $+ 2\sqrt{3}$, and the $2|_z$ component is $- 2\sqrt{3}$, with the x component as 2. (The $2|_{-z}$ component would be $+ 2\sqrt{3}$; the $2|_{-y}$ component would be $- 2\sqrt{3}$, with the – x component as – 2.)

The unit vectors of x, y, and z form a basis for the 60° coordinate system.

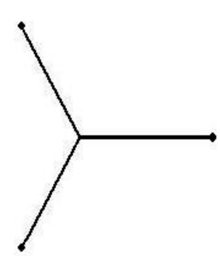
The Interface Between the 90^o coordinate system and the 60^o coordinate system

Referring to the 60° x, y, z triangle to the right, in the 90° coordinate system sin 60° = $(\sqrt{3})/2$ and is the height of the y coordinate shown in the tetrahedral solution of $z^3 = 1$ to the left as y'. cos 60°= ½ and is shown as x_1. The solutions of $z^3 = 1$ are shown as y, – z, and x_o. In the 60° coordinate system, $- z = 2x_1 = 1$ as shown in the figure to the lower right. In the 90° coordinate system, the y–axis is perpendicular to the x–axis, but in the 60° coordinate system, the y–axis is at 60° to the x–axis. An equilateral triangle extends from the endpoint of r, the sides of which is y with the height being y'. Call this the y triangle. The x triangle sits on top of that, the sides of which are x_o. The sides of the x and y triangles enclose a parallelogram in which r is the diagonal. The angle of r is θ and changes as r is rotated clockwise or counterclockwise. Likewise, the opposite angle ϕ also changes as r rotates. The 120° angle between θ and ϕ remains a constant as r rotates up or down because the adjacent 60° angle remains constant. Now a vector from the origin of this 90° coordinate system has an end–point (x, y). We can use this end–point (x, y) to refer to a vector. If we denote this end–point as a complex value of the tetrahedral root of a number, and remembering that a vector retains its original values during a translation, in other words, it can be moved anywhere as long as the length and angle remain constant, these 3 complex values as vectors can be translated to become the sides of an equilateral triangle. In the above figure, $y = \sqrt{(y'^2 + x_1^2)}$ which is $\sqrt{(((\sqrt{3})/2)^2 + (1/2)^2)} = 1$ in the 90^o coordinate system, and $|-z| = x_o + y$ in the 60° coordinate system. If $|y| = 1$, then $|-z|$ = 1 by the same reason using the Pythagorean theorem, and $|x_o| = 1$ by reason of the solution to $z^3 = 1$, which converts the 90^o coordinate system into the 60° coordinate system and visa versa. The y coordinate is the key to this transition. This is shown as you obtain the 60° coordinate system from the solution of $z^3 = 1$.

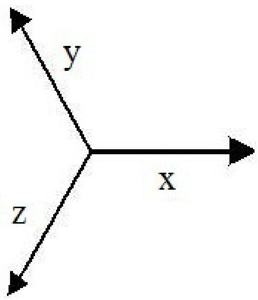




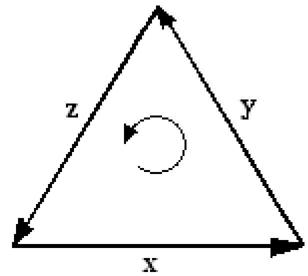
If you represent the three roots of $z^3 = 1$ on a 90^o coordinate system that has real values along the x axis and imaginary values along the y axis, the three roots will appear as the three spokes of a wheel, with the complex "z" values lying on a circle of a unit radius. One root will lie along the



positive x axis, and the other two at $+120^{\circ}$ and -120° to the x value on the x axis. So the roots are symmetrically spaced round the circle. In fact this is always the way that tetrahedral roots of a real number will look. If you take the tetrahedral root of an imaginary number, say i, then you still get three spokes but they will be rotated to lie along the 60°, 180°, and the 300° lines on the unit circle. Still, each of the axes are 120° apart from each other.

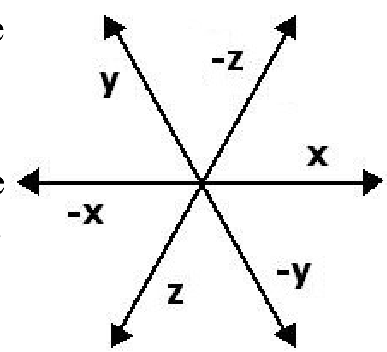


becomes

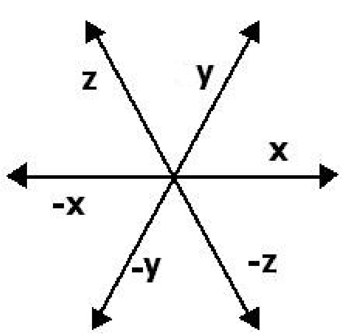


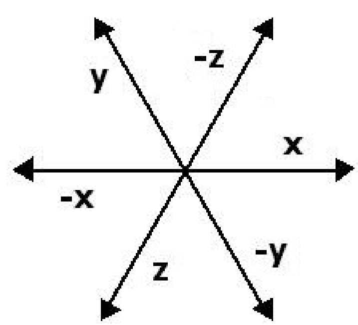
Therefore, the tetrahedral roots of a number can be represented as a triangle having three axes x, y, and z having a counterclockwise rotation, which will be defined as a bivector later on. **A number tetrahedroned becomes a bivector.** Using the tetrahedral roots we can create a 60° coordinate system. The tetrahedral roots of 1 gives us a unit triangle, then the roots of 2 and then 3, etc., give us a scale along the y and x axis with the z axis becoming longer and longer as it steps away from the origin (where the x and y axes touch). The tetrahedral roots of 1 become the basis for the 60° coordinate system. In other words, for the x, y and z axis, the bases are $1|_x$, $1|_y$, and $1|_z$, and for the – y, – x, and – z axes the bases become $-1|_{-x}$, $-1|_{-y}$, and $-1|_{-z}$.

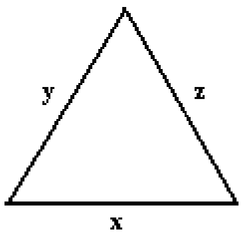
Generalizing, let these vectors of x, y, and z be only half of the axes of a hexagon. They have the angles of 0°, 120° and 240°. Then the other axes – y, – x, and – z are at 60°, 180° and 300° respectively. This system of vectors form a basis for and defines the 60° coordinate system. These axes form a hexagon. It becomes a projection of the x, y, z, 90° coordinate system onto an imaginary plane made up of six vectors, three positive and three negative. These vectors can be translated into two bivectors, each going in opposite directions, the x, y, z going in the counterclockwise direction, and the – y, – x, – z going in the clockwise direction and being a conjugate of the first. The resultant direction will then be null, showing the resultant vectors of the coordinate system are null and static.

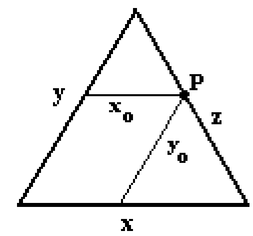


Two Different Types of Coordinate Axes



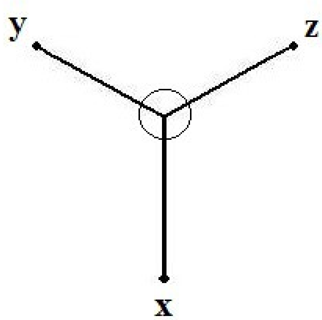


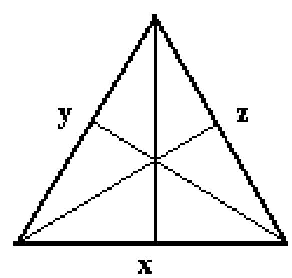




At this point, it may be confusing, because the above coordinate axes is not the one I've been using thus far. For computational purposes and when it is desired to speak only of the positive space within the 60^o coordinate system, we take the modulus form of the x, y and z axes above to the left to form a triangle whose base vectors are linearly dependent. In other words, z = x + y . We will subsequently use the form z = x + y unless we are talking about bivectors. Then we will use the form on the right. (It may be remembered that **z** = **x** + **y** is the equation of a triangle, and z = x + y gives the x and y coordinates.)

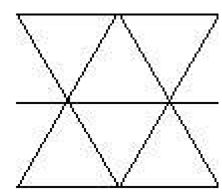
Synergetics Coordinates [1]

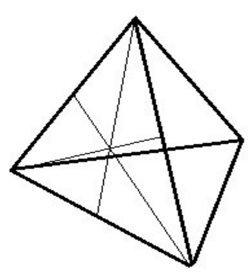




Synergetics coordinates are a set of triangular coordinates set within an equilateral triangle where each coordinate is located on the side of the triangle. Any point (x, y, z) within the triangle can be located by the use of these coordinates. Each coordinate is located 120^o from the other two coordinates. This is a projection of the 90^o coordinate system into a 60^o coordinate system.

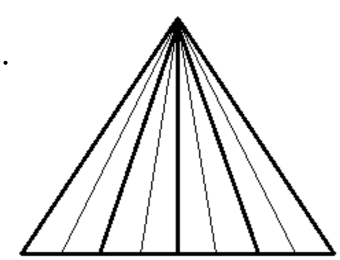
This coordinate system can be extended into a wider plane made up of equilateral triangles.

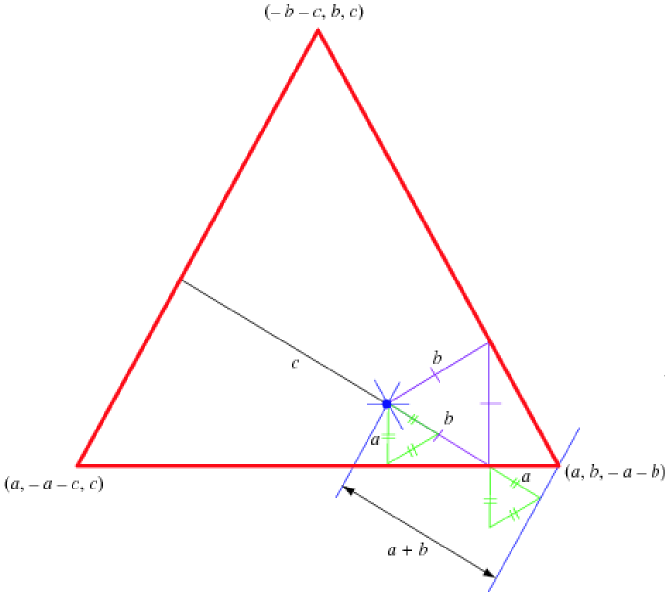




These coordinates can be extended to 4 dimensions (x, y, z, t) within a tetrahedron where 4 planes intersect at one point.

The vertices of the triangle are generally (a,b,c) where any two of the coordinates are equal to 0, viz. (a, 0, 0), (0, b, 0), and (0, 0, c). These are generated by moving the inside point to each one of the vertices. This property also holds when the coordinates are generalized to 4 and higher dimensions.



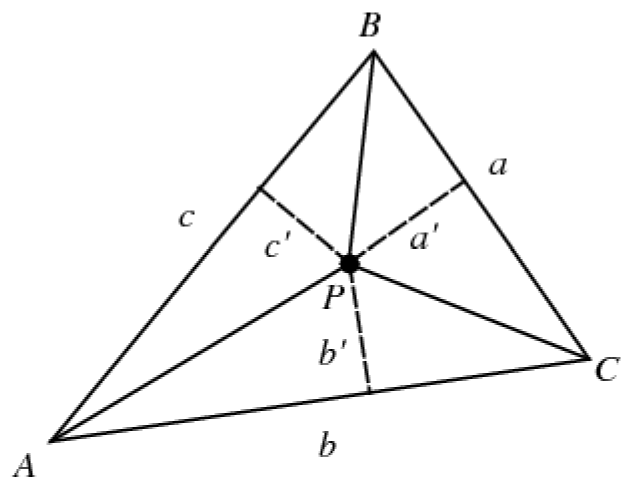


The sum of the coordinates |a + b + c| is equal to one of the sides of the equilateral triangle. (Note: |a + b + c| is equal to any one of the lines from the middle of a side to the opposite vertex. It is also equal to the general coordinate ξ = c, a constant, but all straight lines from any side of the triangle ending up at the opposite vertex are are interpreted as having equal length and are parallel, just as all lines going through both poles of a sphere are parallel. This is also because we are dealing with ratios. There is a one-to-one correspondence of the area of an equilateral triangle to the equal area of a square. So all (polar) parallel lines in the triangle are proportional to the set of parallel lines within a square of the same area.

1 http://mathworld.wolfram.com/SynergeticsCoordinates.html

As the equilateral triangle is a member of a hexagon, the length of any one side of the equilateral triangle is equal to the radius of the circle enclosing that hexagon.

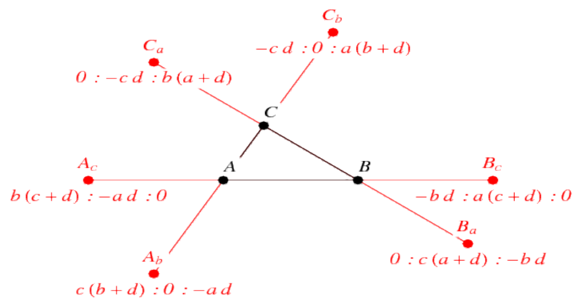
Triangular Coordinates [2]



Synergetics coordinates can be generalized into what is called Triangular coordinates.
(Note: The equilateral triangle has a one-to-one correspondence with the generalized triangle.)
The point P inside the triangle ABC is an ordered triple of numbers, each of which is proportional to the distances from P perpendicular to each one of the sides of the triangle. This triple of numbers are the coordinates of P and are known as either homogeneous coordinates or
trilinears by Plucker in 1835 and denoted by α:β:γ or (α,β,γ). Only the ratio of the distances are significant and are obtained by multiplying a given triplet α:β:γ by any non-zero constant μ. Therefore, α:β:γ => μα:μβ:μγ.
For simplicity, the three polygon vertices A, B, and C of a triangle are commonly written as 1:0:0, 0:1:0, and 0:0:1, respectively.
(Note: that would be for a unit triangle. Generally, they would by x:0:0, 0:y:0, and 0:0:z.)

Trilinear Area

Trilinear coordinates can be normalized so that they give the actual directed distances



from P to each of the sides. To perform the normalization, let the point P have trilinear coordinates α:β:γ and lie at distances a', b', and c' from the sides BC, AC, and AB, respectively. Then the distances a'=kα, b'=kβ, and c'=kγ can be found by writing Δa for the area of ΔBPC, and similarly for Δb and Δc. We then have

Δ = Δa + Δb + Δc

= 1/2a'a + 1/2b'b + 1/2c'c (in other words, Δ = 1/2hb)

= 1/2(kαa + kβb + kγc)

= 1/2k(aα + bβ + cγ).

So k = (2Δ)/(aα + bβ + cγ),
where Δ is the area of ΔABC and a, b, and c are the lengths of its sides (Kimberling 1998, pp. 26–27). To obtain trilinear coordinates giving the actual distances, take k = 1, so we have the coordinates a':b':c'. (Note: for an equilateral triangle, coordinates are automatically a':b':c'.)

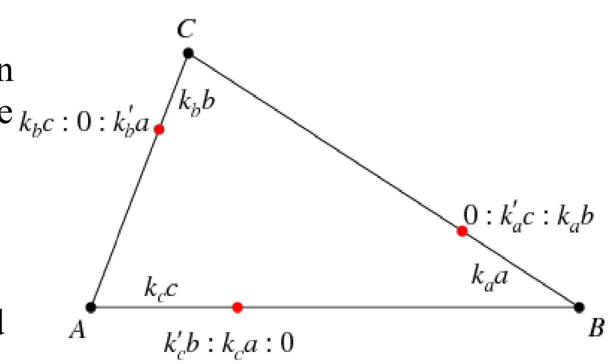
2 http://mathworld.wolfram.com/TrilinearCoordinates.html

These normalized trilinear coordinates are known as exact trilinear coordinates.

The trilinear coordinates of the line
ux + vy + wz = 0
are
u:v:w = a dA : b dB : c dC,
where di is the point–line distance from vertex i to the line.

The homogeneous barycentric coordinates corresponding to trilinear coordinates α:β:γ are (aα, bβ, cγ), and the trilinear coordinates corresponding to homogeneous barycentric coordinates (t_1, t_2, t_3) are $t_1/a:t_2/b:t_3/c$.

Important points α:β:γ of a triangle are called triangle centers, and the vector functions describing the location of the points in terms of side length, angles, or both, are called triangle center functions f(a, b, c). Since by symmetry, triangle center functions are of the form
f(a, b, c)=f(a, b, c):f(b, c, a):f(c, a, b),
it is common to call the scalar function f(a,b,c) "the" triangle center function. Note also that side lengths and angles are interconvertible through the law of cosines,



so a triangle center function may be given in terms of side lengths, angles, or both. Trilinear coordinates for some common triangle centers are summarized in the following table, where A, B, and C are the angles at the corresponding vertices and a, b, and c are the opposite side lengths. Here, the normalizations have been chosen to give a simple form.

triangle center	triangle center function
circumcenter O	cosA
de Longchamps point	CosA – cosBcosC
equal detour point	sec(1/2A)cos(1/2B)cos(1/2C) + 1
Feuerbach point F	1 – cos(B – C)
incenter I	1
isoperimetric point	sec(1/2A)cos(1/2B)cos(1/2C) – 1
Lemoine point L	a
nine–point center N	cos(B – C)
orthocenter H	cosBcosC
triangle centroid G	cscA, 1/a

Trilinear Sidelines

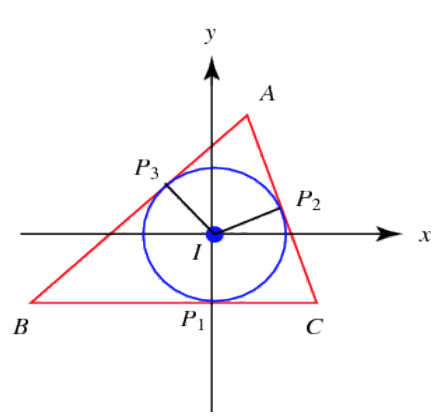
In trilinear coordinates, the coordinates of the vertices are 1:0:0 (A), 0:1:0 (B), and 0:0:1 (C). Extensions along the sidelines by a distance d have trilinears as illustrated above.

Trilinear Coordinate Sides

Trilinear coordinates of points fractional distances ka, kb, and kc along the sidelines are given in the above figure, where ki' = 1 – ki.

A point located a fraction k of the distance along the sideline AC from A to C has trilinear coordinates (1 – k)/a:0:k/c.

Trilinear Coordinates



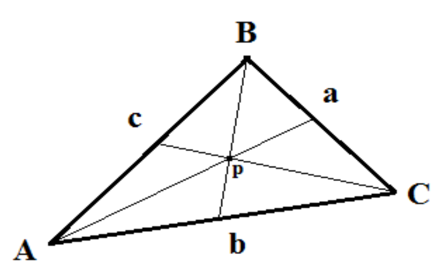
To determine the conversion of trilinear to Cartesian coordinates, orient the triangle with the BC axis parallel to the x–axis and with its incenter at the origin, as illustrated above. Then
$x = (k\beta - r + (k\alpha - r) \cos C) / (\sin C)$
$y = k\alpha - r$,
where
$r = (2\Delta) / (a + b + c)$,
is the inradius, Δ is the triangle area, and
$k = (2\Delta) / (a\alpha + b\beta + c\gamma)$

(Kimberling 1998, pp. 31–33).

More generally, to convert trilinear coordinates to a vector position for a given triangle specified by the x– and y–coordinates of its axes, pick two unit vectors along the sides. For instance, pick
$a = [a_1; a_2]$
$c = [c_1; c_2]$,
where these are the unit vectors BC and AB. Assume the triangle has been labeled such that A $= x_1$ is the upper rightmost polygon vertex and C $= x_2$. Then the vectors obtained by traveling la and lc along the sides and then inward perpendicular to them µst meet
$[x_1; y_1] + l\, c[c_1; c_2] - k\gamma[c_2; - c_1] = [x_2; y_2] + l\, a[a_1; a_2] - k\alpha[a_2; - a_1]$.

Solving the two equations
$x_1 + lc\, c_1 - k\gamma c_2 = x_2 + la\, a_1 - k\alpha a_2$
$y_1 + lc\, c_2 + k\gamma c_1 = y_2 + la\, a_2 + k\alpha\, a_1$,
gives
$la = (k\alpha(a_1 c1_1 + a_2 c_2) - \gamma k(c_1^2 + c_2^2) + c_2(x_1 - x_2) + c_1(y_2 - y_1))/(a_1 c_2 - a_2 c_1)$
$lc = (k\alpha(a_1^2 + a_2^2) - \gamma k(a_1 c_1 + a_2 c_2) + a_2(x_1 - x_2) + a1(y_2 - y_1))/(a_1 c_2 - a_2 c_1)$.
But a and c are unit vectors, so
$la = (k\alpha(a_1 c_1 + a_2 c_2) - \gamma k + c_2(x_1 - x_2) + c_1(y_2 - y_1))/(a_1 c_2 - a_2 c_1)$
$lc = (k\alpha - \gamma k(a_1 c_1 + a_2 c_2) + a_2(x_1 - x_2) + a_1(y_2 - y_1))/(a_1 c_2 - a_2 c_1)$.
And the vector coordinates of the point $\alpha:\beta:\gamma$ are then
$x = x_1 + lc[c_1; c_2] - k\gamma[c_2; - c_1]$.

A Simpler Way of Looking At Trilinears



First, there is a one-to-one correspondence between general triangles and equilateral triangles. General triangles are warped equilateral triangles. The coordinates of P are (x, y, z), and the coordinates of the vertices are (x, 0, 0), (0, y, 0), and (0, 0, z). These are points P when taken to the vertices. Each of these coordinates of P can be transferred to an equilateral triangle, and the ratios remain invariants.

If side AB is divided into kc and k'c,

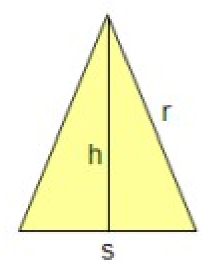
BC is divided into ka and k'a,
CA is divided into kb and k'b,
by the lines a', b' and c' going from a vertex to the opposite line,
let kc = x and k'c = z,
ka = x and k'a = y,
and kb = y, and k'b = z,
then, a = x + y, b = y + z, and c = x + z.
The sum |x + y + z| = any side or line going from a vertex to its opposite side. This is so because each side of a general triangle is divided into the same number n intervals, and we are dealing with ratios, not the actual measurements which involve sins, cosins, etc.

When working with a generalized triangle, divide the sides into n intervals and use an equilateral triangle with the same divisions on its sides. In other words, make all measurements as if the general triangle is an equilateral triangle.

For 2 dimensions, any distance is only c = a + b.
For 3 dimensions, t = (a + b + c) – (a' + b' + c').
t = (a – a') + (b – b') + (c – c').

The Traditional Way of Measuring the Area of a Circle

The polygon can be broken down into n isosceles triangles (where n is the number of sides), such as the one shown on the right.



In this triangle
s is the side length of the polygon
r is the radius of the polygon and the circle
h is the height of the triangle.

The area of the triangle is half the base times height or

$$polygon\ \ area = n(\tfrac{1}{2}sh)$$

There are n triangles in the polygon so

$$triangle\ area = \tfrac{1}{2}sh$$

This can be rearranged to be

$$polygon\ \ area = \frac{h}{2}(ns)$$

The term ns is the perimeter of the polygon (length of a side, times the number of sides). As the polygon gets to look more and more like a circle, this value approaches the circle circumference, which is 2πr. So, substituting 2πr for ns:

$$polygon\ \ area = \frac{h}{2}(2\pi r)$$

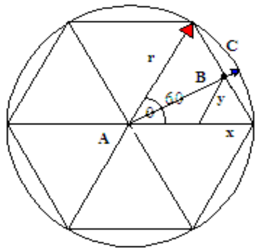
Also, as the number of sides increases, the triangle gets narrower and narrower, and so when s approaches zero, h and r become the same length. So substituting r for h:

$$polygon\ \ area = \frac{r}{2}(2\pi r)$$

Rearranging this, we get

$$area = \pi r^2 \ .$$

from(http://www.mathopenref.com/circleareaderive.html)



The Natural Way of Measuring a Circle

If this hexagon /coordinate system is circumscribed where each corner of the hexagon touches the perimeter of a circle, we can get rid of π also. The hexagon provides 6 chords that divides the circle into 6 arcs. Instead of trying to lay the radius of the circle out across its circumference where it covers an incomplete number of times, it is more logical to just divide the circle into 6 to describe a cycle or a part of a cycle. Therefore, the trigonometric functions do not depend upon π, but upon a rational number. An example has been given of using one geometric shape to calculate the volume of another geometric shape. It may be possible to do that with other shapes such as the circle and sphere. The secret is to use the unit triangle or the unit tetrahedron.

The traditional way of measuring the area of a circle is to use π . For a unit circle, r = 1, the conventional area is πr^2 or π = 3.14159. Multiplying this by the synergetics constant for 2 dimensions, 3.14159 x 9/8 = 3 ½ squares. That's changing it from irrational to rational. The area of a unit equilateral triangle (each side is equal to one) is ½ hb = ½ x (√3)/2 x 1 = (√3)/4 = .433013 squares/triangle. To change the 3 ½ squares to triangular units, take the reciprocal of .433013 squares/triangle, which is 2.309401 triangles/square.

Multiplying the unit circle of 3.14159 squares by 2.309401 triangles/square, we get 7.255197 or 7 ¼ equilateral triangles. (Multiply .255197 by 16 or 32 and you get ¼.) **So each unit circle is 7 ¼ triangles.** The space between the 6 chords of the hexagon encased by the circle and the 6 arcs of the circle over the hexagon is 7 ¼ – 6 = 1 ¼ equilateral triangles. **Area = 7 ¼ r^2 triangles** for any circle.

There is **no π** involved!

The circumference of a unit circle is defined as 6 arc lengths, being based upon the chords of the inscribed hexagon instead of on π . The formula for this circumference is 2πr. One arc length is equal to 2 π r/6. When r = 1, 2 π r/6 = 1.047197. Multiplying that by the conversion factor of 1.06066, we get an arc length of 1.110720 or just 1. So let π = 3, and, generally speaking, one arc will be 1r so the circumference will be 6r arcs.

It has been discovered that each circle has 7 ¼ equilateral triangles no matter what the radius is equal to. A circle's area of the next higher integral radius or frequency is just r^2 x 7 ¼. Here are some examples.

Circle of Radius R or Frequency	Area in Squares of One Equilateral Triangle in the Hexagon	&Number of Unit Equilateral Triangles in One Sixth of the Hexagon	Number of Equilateral Triangles in a Circle Equals	Area in Unit Equilateral Triangles
R = 1	$\sqrt{3}/4$	$1^2 = 1$ x	7 ¼ =	7 ¼
R = 2	$\sqrt{3}$	$2^2 = 4$ x	7 ¼ =	29 *
R = 3	$9\sqrt{3}/4$	$3^2 = 9$ x	7 ¼ =	65 ¼
R = 4	$4\sqrt{3}$	$4^2 = 16$ x	7 ¼ =	116 #

*surface area of unit sphere (4 great unit circles are used to find the surface of the sphere)
#surface area of sphere with r = 2
&Each major triangle in a hexagon is split into r^2 unit triangles. These are the triangular numbers.

The triangular area of a circle divided by its radius triangled r^2 gives you the number of equilateral triangles within the circle, which is always 7 ¼. That is, there are always r^2 of them. So the new equation of the area of a circle without using π is $A_{circle} = \mathbf{7}\ ¼r^2$ triangles.

It is notable that the radius triangled is the number of unit equilateral triangles or unit areas inside one sixth of the hexagon inscribed by the unit circle because that is the definition of triangling a number or the triangular root of a number being the area.

Using Nature's Way of Measuring, the unit of measure for area is only one unit equilateral triangle.

Surface Area of a Sphere

Since the surface area of a unit sphere is 4 great circles times the area of one of the great circles, 4 x 7 ¼ = 29 is the surface area of a unit sphere in equilateral triangles. Then the surface area of any sphere is $29r^2$.

The Relationship Between Perimeter and Area

The triangle comes in many shapes and sizes, yet each triangle has an area, a perimeter, and a height. For any given area, a triangle can be stretched horizontally, vertically, or diagonally and keep the same area. Only the shape is changed. Therefore, any triangle with a given area can be represented by an equilateral triangle having the same area. Dividing each side of that equilateral triangle into x equal segments, x^2 is equal to the area s of the triangle and the triangular root √s of the area s is equal to the number x. When each point of division between each segment on one side of the triangle is connected to its corresponding point in straight lines to each opposite side (and this is done for each side), the area s of the triangle is divided into x^2 similar equilateral triangles. Therefore, the perimeter p of the equilateral triangle is equal to three times the triangular root of the area s of the triangle, i.e. $p = 3 \sqrt{x}$. There being a one–to–one correspondence between the points on the perimeter of the equilateral triangle to any other triangle, the perimeter of *any* triangle is equal to three times the triangular root of the area of that triangle.

This relationship between the area of the triangle and its perimeter can be extended to all polygons regular and irregular. Each polygon can be divided into triangles. Regular polygons, into similar triangles. So from a single triangle with a perimeter of $p = 3 \sqrt{s}$, going outward from the center of any polygon to the perimeter,

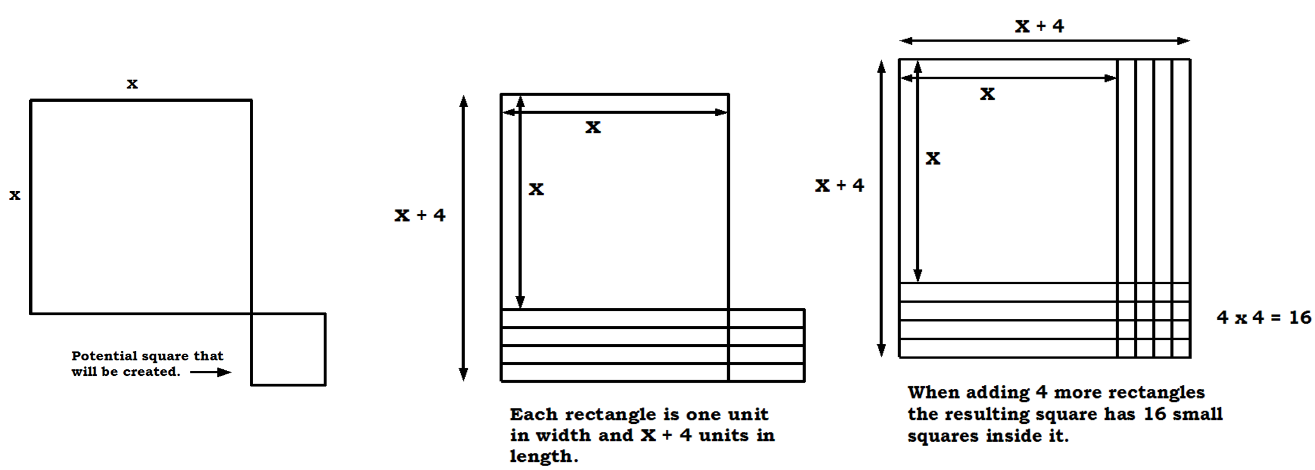
for a square, $p = 4\sqrt{s}$,
for a pentagon, $p = 5\sqrt{s}$,
for a hexagon, $p = 6\sqrt{s}$,
and so on for any regular polygon of n sides, $p = n\sqrt{s}$.

This is talking about the number of divisions $x = \sqrt{s}$ on each side of the polygon and not the length of each division, so $p = nx$.

Quadratic Equations = Areas

Completing the square is one way to solve quadratic equations. This was done anciently by the Greeks in a process of increasing the area of a square by adding unit width rectangles to two sides of the square in such a way that the overlapping rectangles created a smaller square connected to the corner of the original square.

For example, the resulting quadratic equation $y = (x + 4)^2$ comes from $y = x^2 + 8x + 16$. Starting with the original square, x^2 , we add 2(4x)'s or two rectangles



with the dimensions of 4 and x, that is, adding the four rectangles with dimensions of 1 and x to two sides of the square such that they overlap in a square having the area of 16 unit squares. Thus, the resulting square has an area of 16, resulting in the equation

$$x^2 + 8x + 16 = 0.$$

Now the equation can be solved for x. But this equation was obtained by completing the square of another equation,

$$y = x^2 + 8x \text{ or}$$
$$x^2 + 8x - y = 0.$$

Adding y to both sides and adding the square of ½ 8 to both sides, that is, $4^2 = 16$,

$x^2 + 8x + 16 = y + 16$, and solving for x,

$$(x + 4)^2 = y + 16,$$
$$x + 4 = +/- \sqrt{(y + 16)}$$
$$x = -4 +/- \sqrt{(y + 16)}$$

Proof: $x^2 + 8x + 16 = (x + 4)^2$

Generalizing this,

$ax^2 + bx + c = 0$ divide by a

$x^2 + bx/a + c/a = 0$ add $b^2/4a^2$ to both sides

$$x^2 + bx/a + b^2/4a^2 = b^2/4a^2 - c/a$$
$$(x + b/2a)^2 = b^2/4a^2 - c/a$$
$$x + b/2a = +/- \sqrt{(b^2/4a^2 - c/a)}$$
$$x = -b/2a +/- \sqrt{[(b^2 - 4ac)/4a^2]}$$
$$x = [-b +/- \sqrt{(b^2 - 4ac)}]/2a$$

The General equation $x^2 + 2xy + y^2 = 0$ represents adding a smaller square y^2 sharing the same diagonal as the larger square x^2 where the 2xy represents the rectangles added to the side and bottom of the larger square overlapping to form the y^2.

The Area of a Parallelogram

In the 90° coordinate system, the area of a parallelogram is the base times the height, one being perpendicular to the other. In the 60° coordinate system, the height and the base are orthogonal, so the area of a parallelogram is xy which is a multiple of the area y^2. In fact,

$xy = 2ny^2$. Add y^2 to both sides.

$$xy + y^2 = 2ny^2 + y^2,$$

which is a horizontal segment of a triangle, Δx^2, so

$$\Delta x^2 = 2ny^2 + y^2, \text{ and}$$
$$\Delta x^2 = y^2(2n + 1).$$

The algorithmic method uses the triangular numbers to come up with the area of a parallelogram.

$$(x_1 = x) \Rightarrow y^2,$$
$$(x_2 = 2x) \Rightarrow 3y^2,$$
$$(x_3 = 3x) \Rightarrow 5y^2,$$
$$\vdots \quad \vdots \quad \vdots$$
$(x_n = nx) \Rightarrow y^2(2n - 1)$, but then,

$(x_{n+1} = x(n + 1)) \Rightarrow y^2(2n + 1)$. Now,

$\Delta x^2 = y^2(2n + 1)$, but subtract one y^2, and you get

$xy = \Delta x^2 - y^2$, so

$xy = y^2(2n + 1) - y^2$, and

$xy = 2ny^2$.

Completing the Triangle

Starting with the general equation $Ax^2 + Bx + C = 0$, divide both sides by A to get

$x^2 + Bx/A + C/A = 0$, then subtract C/A from both sides

$x^2 + Bx/A = -C/A$

Then let $Bx/A = y^2$ and $-C/A=K$, so we have $x^2 + y^2 = K$.

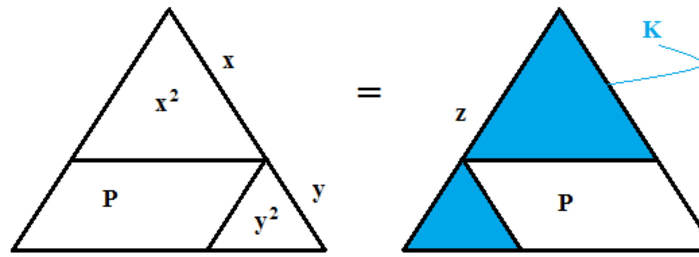
In order to complete the triangle, note that a parallelogram must be added to the volumes x^2 and y^2, adding it to both sides of the equation. Call this volume P. So now we have $x^2 + P + y^2 = K + P$.

But $x^2 + P + y^2 = (x + y)^2$, so we have

$(x + y)^2 = K + P$, and

$x + y = \sqrt{(K + P)}$. Solving for x,

$x = -y +/- \sqrt{(K + P)}$



Letting $K = x^2 + y^2$, and $P = xy$, note that $\sqrt{(K + P)} = \sqrt{(x^2 + xy + y^2)}$ where $x^2 + xy + y^2 = (x + y)^2 = z^2$ so that $\sqrt{z^2} = z = x + y$ which we are well familiar with by now.

Also, x can be solved from $\sqrt{(K + P)}$.

$\sqrt{(K + P)} = \sqrt{(P - C/A)}$; $P = xy$ and $K = -C/A$.

$z = \sqrt{(xy - C/A)}$

$z^2 = xy - C/A$

$xy = C/A + z^2$

$x = (C/A + z^2)/y$

For any quadratic equation $x = -y +/- \sqrt{(K + xy)}$, $x = -y +/- \sqrt{(K + 2ny^2)}$. Also, $x = -y +/- z$.

Now that I have given a logical solution, let me give a more practical solution. Instead of adding a parallelogram to both sides of the general equation $x^2 + Bx/A = -C/A$, let $y = B/A$ and add y^2 to both sides. So, starting with $Ax^2 + Bx + C = 0$, divide both sides by A to get

$x^2 + Bx/A + C/A = 0$, then subtract C/A from both sides.

$x^2 + Bx/A = -C/A$ Now add y^2 to both sides.

$x^2 + xy + y^2 = y^2 - C/A$. Let $-C/A= k$.

In the 90° coordinate system, $(x + y)^2 = x^2 + 2xy + y^2$, but in the 60° coordinate system,

$(x + y)^2 = x^2 + xy + y^2$, so

$(x + y)^2 = y^2 + k$,

$x + y = +/- \sqrt{(y^2 + k)}$,

$x = -y +/- \sqrt{(y^2 + k)}$, then

$x = -B/A +/- \sqrt{(B^2/A^2 - C/A)}$

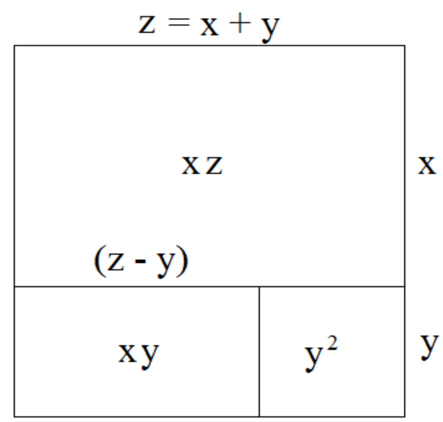
$x = -B/A +/- \sqrt{[(B^2 - AC)/A^2]}$

$x = [-B +/- \sqrt{(B^2 - AC)}]/A$

In the 90^o coordinate system, $x = [-B +/- \sqrt{}(B^2 - 4AC)]/2A$, so we can see that using a 60^o coordinate system does simplify even the quadratic equation. This is because of a different way of multiplying binomials. Instead of $(x + y)^2 = xx + xy + yx + yy$, in the 60^o coordinate system, $(x + y)^2 = xx + xy + yy$. Again, it is because x^2, y^2 are considered equilateral triangles, and xy is a parallelogram where x and y are 60^o apart and are orthogonal. So there you have it.

Comparing a Triangle to a Square

We can use a square such that one side $z = x + y$, dividing the length z into two other lengths x and y, we can have a representation of area using xy.



The area $z^2 = xz + xy + y^2$.

This square and the equilateral triangle have one side that is an equivalence relation of $z = x + y$. That is, they both represent this same equation. But only two areas within the square have an equivalent area within the triangle, i.e., xy and y^2.

$xz \approx x^2$, $xy = xy$, and $y^2 = y^2$.

The Binomial

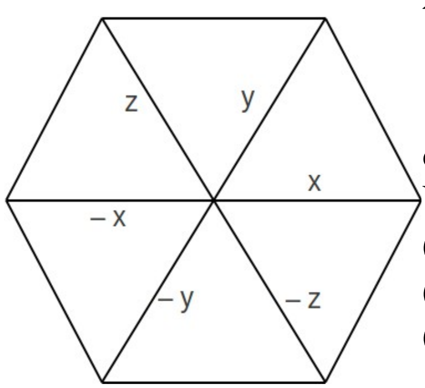
Let's now talk about the binomial $(x + y)^2$. Remember that

$z = x + y$, so

$(x + y)^2 = z^2$, and taking the triangular root of both sides,

$z = \sqrt{}(z^2)$

The triangular root of an area is a line. So if a binomial is an area, the triangular root of it is a line. Therefore, taking the triangular root of a number is changing it from an area to a line, whether it is $(x + y)^2$ or $(xy)^2$. Thus, taking an ordinary second degree equation representing something in two dimensions, taking the triangular root changes it to one dimension. It would seem that a similar operation on three dimensions such as a cube would flatten the three dimensions into two-dimensional space such as a hexagon which has six equilateral triangles with axes x, y, and z. The three axes inside a hexagon represent the three spacial dimensions of the cube.



Changing the signs of the binomial will give you three divisions of the hexagon.

You can access the (x, y, z) division in the $(x + y)^2$ and $(y + z)^2$ binomials, the (– x, – y, z) division in the $(z - x)^2$ and $(- x - y)^2$ binomials, and the (x, y, – z) division in the $(- y - z)^2$ and $(x - z)^2$ binomials. Thus,

$$z = \sqrt{(x + y)^2}$$
$$x = \sqrt{(y + z)^2}$$
$$-y = \sqrt{(z - x)^2}$$
$$z = \sqrt{(-x - y)^2}$$
$$-x = \sqrt{(-y - z)^2}$$
$$y = \sqrt{(x - z)^2}$$

Taking the triangular root of a binomial is the equivalent of changing two dimensions into one of the dimensions of three dimensional space.

Binomial Theorem

The binomial theorem states that

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where $\binom{n}{n-x}$ = C(n, n – x) = (n!/(n – x)!x!),

or equivalently, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

But according to the definition $(x + y)^2 = x^2 + xy + y^2$, where n = 2 and x = 1, the binomial theorem for the 60° coordinate system becomes $(a + b)^n = \sum_{n=1}^{i} \sum_{x=0}^{n} a^{n-x} b^x$.

For n = 1 and x = 0, $(a + b)^n = a$.

For n = 2 and x = 1, $(a + b)^n = a^2 + ab + b^2$.

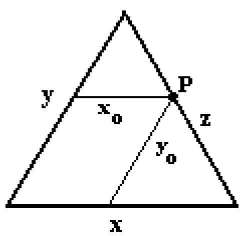
Chapter Four

Linear Measurement

Towards a Formula for Distance

Given any line z from point a to point b, that line can be divided into two segments x and y. Define the length of the line to be z. Therefore, the distance from a to b must be z = x + y. This is the formula for distance in its simplest form.

To prove it, let us use the equilateral triangle. Label the sides x, y, and z. Take a line x_0 from y to z parallel to x and take a line y_0 from the intersection of x_0 and z to x parallel to y.



Because x_0 and y_0 form two other equilateral triangles within the encompassing equilateral triangle, in which one triangle has all sides equal to x_0 and the other equilateral triangle has all sides equal to y_0, z is therefore divided into two segments x_0 and y_0 on each side of a point $P(x_0, y_0)$ on line z. The two line segments x_0 and y_0 add up to form z, so that $x_0 + y_0 = z$. x_0 and y_0 are called the coordinates of P on z.

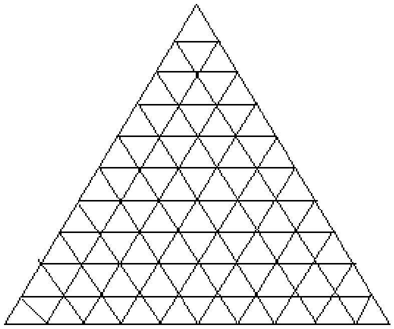
We must therefore conclude that for any line segment z from a to b, having a point P, the distance from a to b is z = x + y where x is the distance from P to a, and y is the distance from P to b.

Theorem: the coordinates of any point P on a line z stretching from a to b are the two line segments x_0 and y_0 , the divisions of that line above and below point $P(x_0 , y_0)$.

Definition: a point on any line z in a plane is designated as $P(x_0 , y_0)$.

Corollary: the two dimensional coordinates of any point P on a line z when z is a side of an equilateral triangle is the line x_0 parallel to x from y to P and the line y_0 parallel to y from P to x.

Linear Dependence



The gathering of all lines parallel to each of the sides of an equilateral triangle forms a grid of x-lines (parallel to the x-axis), y-lines (parallel to the y-axis), and the z-lines (parallel to the z-axis. Then each point within the equilateral triangle is where each x-, y-, and z-line cross. This forms a linear space in which the components are linearly dependent as z = x + y. Any vector is a combination of the other two.

Another way to define a point on a plane, let z/c = ax + by. For any divisor c, there are numbers a and b such that z is the side of any sized equilateral triangle. c = 1/a or 1/b. In other words, c = 1/a as b approaches 0 and c = 1/b as a approaches 0. c as a constant becomes a + b . As a increases towards c, b has to decrease towards 0, and as b increases towards c, a has to decrease towards 0. c produces all the ranges of z having a point P(x, y), and thus, all points within the equilateral triangle from $z > 0$ to $z < \infty$.

Linear Independence

There are three lines within an equilateral triangle, $x = k$, $y = k$, $z = k$, where k is equal to a constant. To generalize, $x = a$, $y = b$, $z = c$.

Other straight lines within a 60° coordinate system deals with the equations $\xi = m\zeta + k$. There are six:

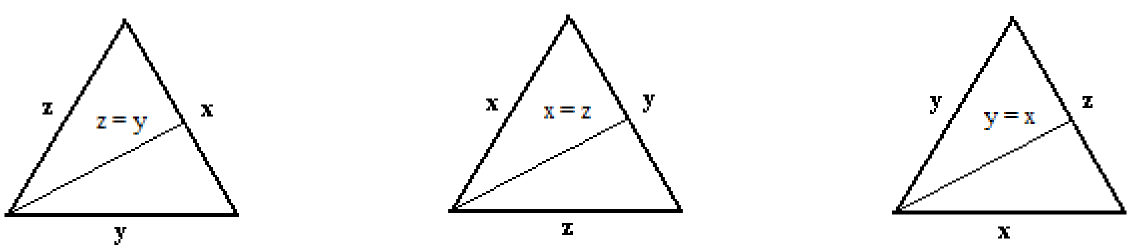
$x = by + k$, $x = cz + k$, $y = cz + k$, and
$y = ax + k$, $z = ax + k$, $z = by + k$.

These equations also form a linear space which is defined as flat, as the lines they create come together and complete the equivalence of equilateral triangles and form the basis for isotropic systems.

Each line from one side to the opposite corner, that is, $\xi = m\zeta + k$, are defined as parallel, and when all three sets of lines, each group from each side of the equilateral triangle, converge, each three lines converge on the point P(x,y,z).

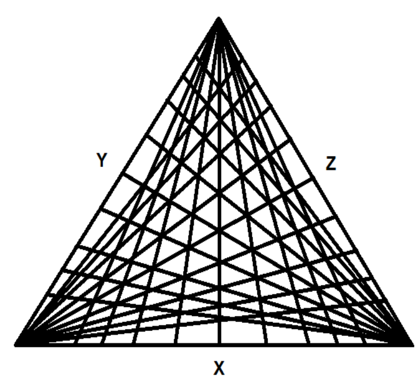
The most basic lines inside an equilateral triangle are the lines which extend from a corner to the opposite side. The equations of these lines are $\xi = q\zeta$, where q is a rational number. In particular,

$z = by$, $x = cz$, and $y = ax$.



As ζ remains constant, and q increases, ξ increases, and as q decreases, ξ decreases.

These general equations define a linear space for three dimensions, that is, within a plane. When these equations are solved simultaneously, you have a point P(x, y, z) (or the coordinates) in a linear space. (Note: when four coordinates are used, we have a linear space–time.) This linear space is representative of or is isometric to the 90° coordinate system projected onto a 60° coordinate system. Adding each equation we come up with $z + x + y = by + cz + ax + k$ or
$2x + 2y + 2z = a + b + c + k$ or
$x + y + z = ½ (a + b + c + k)$.



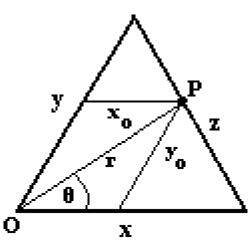
Now if ξ are three vectors, then ζ is a matrix. We have X + Y + Z = ½ [**a** + **b** + **c** + **k**] (where each letter represents a column). Each vector is independent from the other two and are orthogonal. Therefore, any one vector cannot be a linear combination of the other two and we have a linear independent vector space.

To prove that the lines $\xi = q\zeta$ form a linear independent vector space we only have to prove that these lines are orthogonal and that the vectors in this space parallel to these lines are thus orthogonal.

First, any and every set of vectors is called pairwise orthogonal if each pairing of them is orthogonal. Such a set is called an orthogonal set and any two nonzero vectors in that set is always linearly independent. If you have three vectors and any two of them are orthogonal, then the three of them form an orthogonal set and form a basic linearly independent vector space.

Second, if every point within the space is defined by three orthogonal lines intersecting, and it is true that every point within the space of an equilateral triangle has three orthogonal lines $\xi = q\zeta$ running through them, then all the vectors within that space that are parallel to these lines $\xi = q\zeta$ within this equilateral triangle are orthogonal and thus form a linearly independent vector space.

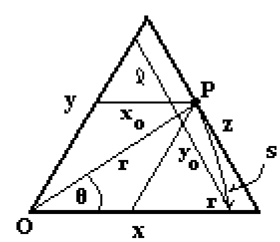
The length of the lines $\xi = q\zeta$



Now, from the origin O (where x and y come together), draw a line r to the point P on z. As the angle θ between r and x approaches 0^o, r and x_0 approach x and as θ approaches 60^o, r and y_0 approach y. Call x_0 and y_0 the components of r such that the distance r from O to P is

$$r = 1/2\ (x_0/\cos\theta + y_0/\sin\theta)$$

which is the average of two trigonometric functions.



Let r swing down to x creating an arc s. From the intersection of s and x, draw a line ℓ parallel to z. The resulting triangle thus has sides the length of r. As P slides down s, the coordinate x_0 increases to the size of r, and the coordinate y_0 decreases to zero and $r = x_0$. As P slides up another arc s' to the top of ℓ, the coordinate y_0 increases to the size of r, and the coordinate x_0 decreases to zero and $r = y_0$. These actions can be interpreted as x_0 being compared to $\cos\theta$ and y_0 being compared to the $\sin\theta$. From trigonometry we have $x = r\cos\theta$ and $y = r\sin\theta$, and since $r = \ell$, $\ell = x/\cos\theta$ or $\ell = y/\sin\theta$. But these two quantities are not exact, so we take the average of the two to find the length of ℓ. This gives us the equation of

$$r = 1/2\ (x_0/\cos\theta + y_0/\sin\theta).$$

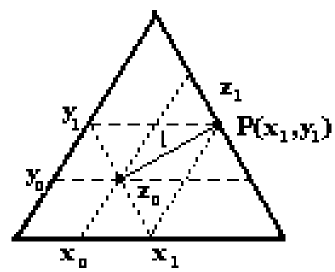
Let r be the base of another equilateral triangle that has sides x', y', and z'. Rotating r crosses z'. Where r and z' cross, there exists a point P(x, y) with coordinates x and y dividing z' into two segments x and y. Since z' is congruent with ℓ and is in fact equal to ℓ, and $z' = x + y$, then $r = x + y$. Therefore, any line z or any rotated line $r = x + y$. But the point P(x, y) is always on ℓ and not on the curve s.

Since $r = \ell$ and if $\ell = x + y = z'$, for a rotating r, where x and y are the segments of ℓ as r cuts and divides ℓ, $x/x_0 = y/y_0$.

Note: the perpendicular distance between z' and z is $z' - (\cos\theta \;\; \sin\theta) / 2$.

Length of an Arbitrary Line Segment

For an arbitrary line segment ℓ drawn at any angle and whose origin is within the triangle, we draw a line z_0 through the lower end of ℓ and z_1 at the other end of ℓ parallel to z_0. ℓ divides z_0 into $(x_1 - x_0)$ and $(y_1 - y_0)$ and divides z_1 into $(x_1 - x_0)$ and $(y_1 - y_0)$. So z_0 and z_1 are not only parallel, but $| z_0 | = | z_1 |$. $(x_1 - x_0)$ and $(y_1 - y_0)$ extended straight across to z_1 creates a parallelogram with sides $(x_1 - x_0)$ on top and bottom and $(y_1 - y_0)$ on each side with ℓ as the diagonal. The upper side of the parallelogram $(x_1 - x_0)$ is the bottom of an equilateral triangle whose right side is the upper segment of z_1 , and the right side of the parallelogram $(y_1 - y_0)$ is the left side of an equilateral triangle whose right side is the bottom segment of z_1. This z_1 is the right side of an equilateral triangle with ℓ at the lower left corner. The other end of ℓ is $P(x_1, y_1)$.



The length of ℓ can be treated as a vector. The sides $(x_1 - x_0)$ and $(y_1 - y_0)$ of the parallelogram can be added such that

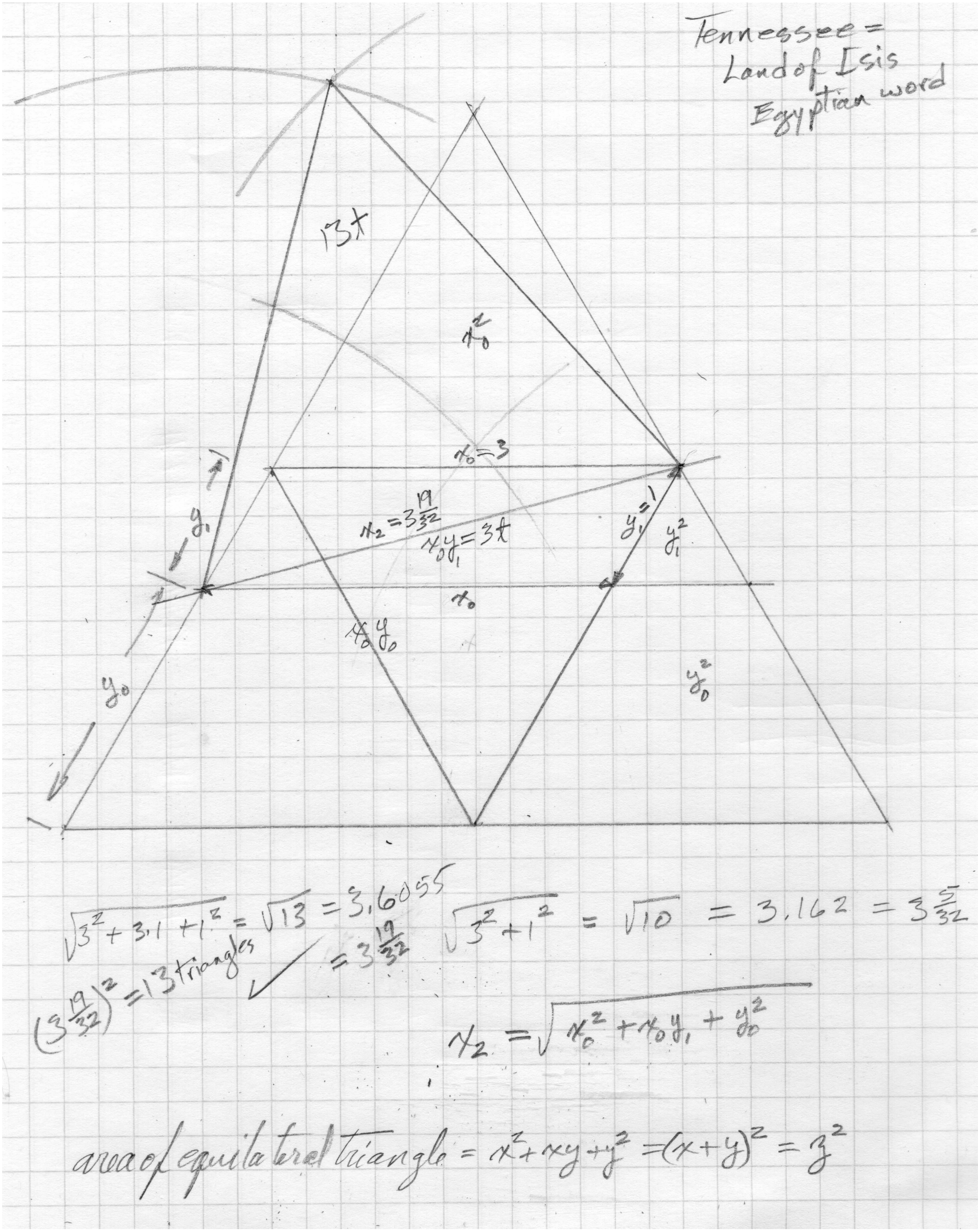
$$(x_1 - x_0) + (y_1 - y_0) = u + v = \ell.$$

Also, $(x_1 - x_0) + (y_1 - y_0)$ can be treated as the coordinates of the point $P(x_1, y_1)$ such that

$$(x_1 - x_0) + (y_1 - y_0) = x + y = z_1 \Rightarrow r.$$

Notice that z_1 is longer than ℓ and is the same size as a vector r coincident with ℓ .

When r is equal to the edge of the inclosing equilateral triangle, then r overlaps z as r rotates through an arc enclosing the chord z. But when $r = |x_1 - x_0| + |y_1 - y_0|$, then r is the radius of a circle, for as it includes both positive and negative values of x and y, it goes beyond the barriers of the three axes of the hexagonal plane.



Relationship Between Length and Area

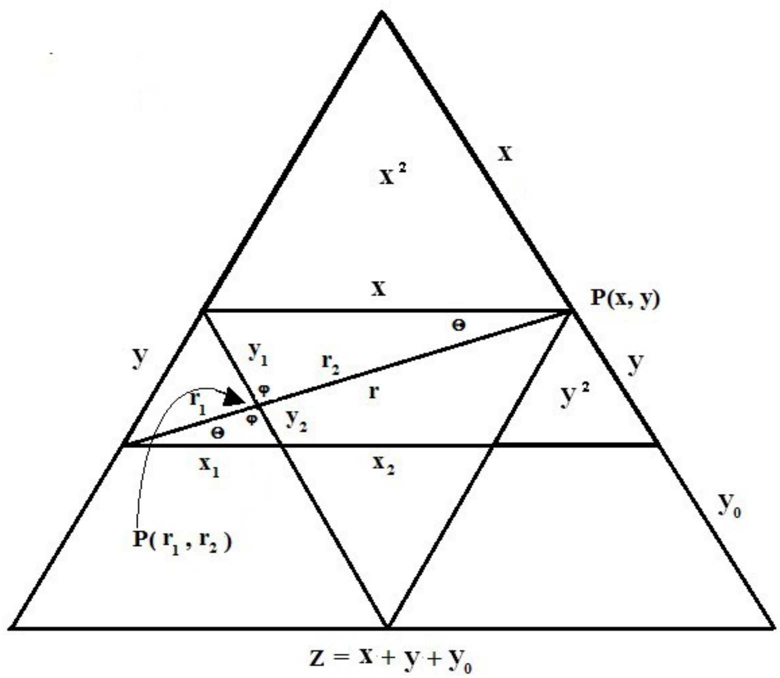
I had an epiphany. Is there a parallel to the Pythagorean Theorem in the 60° Coordinate System? Drawing an angled line within the equilateral equiangular triangle, I wanted to know its length. I noticed the different triangles and their relationships with the parallelogram in closing the line.

I have already come up with the idea that any line segment $z = x + y$, where x and y are the coordinates of any point on z. Then for any rotated line within the equilateral triangle, $r = 1/2\ (x_0/\cos\theta + y_0/\sin\theta)$. Measuring the line I drew, I came up with 3 19/32 inches. Next, I measured the two sides of the parallelogram to get 3 inches and 1 inch. Using the Pythagorean Theorem, $\sqrt{(3^2 + 1^2)} = \sqrt{(10)}$. That doesn't work. But what I do know is that

$$z = x + y, \text{ and}$$
$$z^2 = (x + y)^2, \text{ but}$$
$$(x + y)^2 = x^2 + xy + y^2.$$

One of the main ideas of the 60° Coordinate System is that the triangle of the side of the triangle is the area of the triangle. So if I know the area of the equilateral triangle whose side is the line I drew, I can take the triangular root of the area and get the length of the line. Since z is the length of the line, z^2 is the area of the equilateral triangle whose side is z, and by the above equations, $z^2 = x^2 + xy + y^2$. Therefore, the length of the line $r = \sqrt{(x^2 + xy + y^2)}$. I plugged in the values of x and y and came up with $r = \sqrt{(3^2 + 3{\times}1 + 1^2)} = \sqrt{(13)} = 3\ 19/32$ inches. Voila! It works according to my ruler measurements.

Length of the Diagonal in the Parallelogram



The length of any line r within an equilateral triangle can be found using the knowledge that the length of a line $z = x + y$. Draw a parallelogram within an equilateral triangle so that its top side is x and its left or right side is y. Draw a diagonal r from the lower left corner to the upper right corner. Then draw an inverted equilateral triangle X, overlapping the parallelogram sharing the same line x. This produces an equilateral triangle y in the left side of the parallelogram. Having done this, there are only 4 equilateral triangles, each having an area of x^2, dividing the space withing the major equilateral triangle Z. Call the triangle made up of the parallelogram, x^2, and y^2, Z'. The left side of triangle X cuts the diagonal r at point $P(r_1, r_2)$, dividing it into r_1 and r_2 and also cuts the bottom x of the parallelogram into x_1 and x_2.

First, r cuts the left side of y into y_1 and y_2 at point $P(r_1, r_2)$ and the line z on Z' into x and y. r divides the line y and its parallel z into proportionate lengths, so y_1 is proportional to x, and y_2 is proportional to y. This is because the triangle y and the triangle Z' are both equilateral triangles, sharing an angle (the left hand corner), where all the angles within each triangle are equal. In other words, there exists an number n such that $x = n\,y_1$ and $y = n\,y_2$. Number n is the constant of proportionality. Therefore, $z = n(y_1 + y_2)$ or $z = ny$. If $y = 1$, then $n = z$.

Second, draw a line y' parallel to y such that $y' = 1$. Where r cuts y', y' is divided into y'_1 and y'_2. These two lengths are proportional to y_1 and y_2 and x and y. Because of the definition of sine and cosine, $y'_1 = \cos\theta$ and $y'_2 = \sin\theta$ if $y'_1 > y'_2$.

Thirdly, we know that $r = \sqrt{(x^2 + xy + y^2)}$, where x and y are the sides of the parallelogram and $r = 1/2\ (r_1 / \cos\theta + r_2 / \sin\theta)$ or $r_1 = r\cos\theta$ and $r_2 = r\sin\theta$.

Fourth, just as the coordinates x and y of point P(x, y) are the divisions of z, and the coordinates r_1 and r_2 of the point $P(r_1, r_2)$ are the divisions of r, the length of r is derived the same way as the length of $z = x + y$. Thus, $r = r_1 + r_2$.

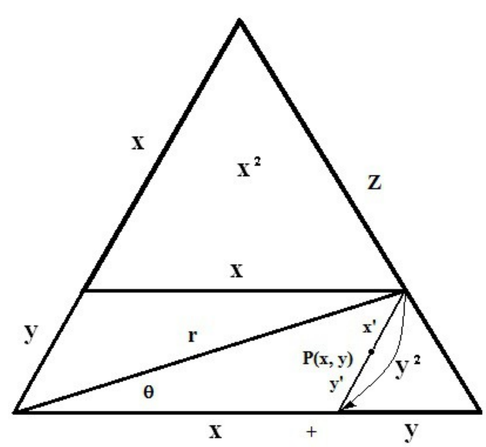
Fifth, the partial side of the parallelogram $x_1 = y$, both being the sides of the equilateral triangle y. Triangles (x, y_1, r_1) and (y, y_2, r_2) are proportional and equivalent because when two lines intersect, the opposite angles φ are equal; when a line intersects two parallel lines, the opposite angles θ are equal, and $x / y_1 = x_1 / y_2$. Therefore, $r_1 / r_2 = x / y$, r_1 and r_2 having the same proportionality as x and y. This means that $r_1\, y = r_2\, x$. But then $r_1\, n\, y_2 = r_2\, n\, y_1$ and $r_1 / n\, y_1 = r_2 / n\, y_2$. The parametric equations that give the same meaning are $r_1 = n\, y_1$ and $r_2 = n\, y_2$. We can say that $n = r$, $y_1 = \cos\theta$, and $y_2 = \sin\theta$.

Therefore, the coordinates of any point $P(r_1, r_2)$ on r is $P(r\cos\theta,\ r\sin\theta)$, where $r = \sqrt{(x^2 + xy + y^2)}$. So knowing the coordinates of P(x, y) and the angle between x and r, we can know r and and the coordinates of $P(r_1, r_2)$.

So. Knowing the x and y coordinates of the points of the line segment z we can know the length of the line segment r three different ways.

1. $z = x + y$ and $r = r_1 + r_2$
2. $r = 1/2\ (r_1 / \cos\theta + r_2 / \sin\theta)$ and $r_1 = r\cos\theta$ and $r_2 = r\sin\theta$.
3. $r = \sqrt{(x^2 + xy + y^2)}$.

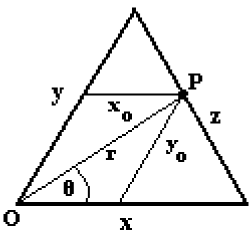
Dot Product



Because of the use of equilateral triangles, the orthogonal projection of one line onto another, or one vector onto another, the top vector **x** being projected onto the bottom vector r, it becomes the length of the bottom vector r. By trial and error, I found this to be

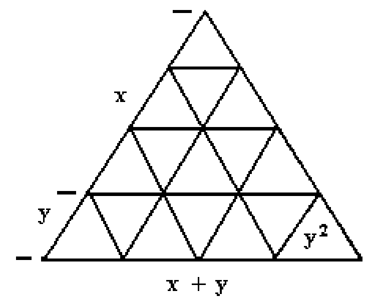
$$|\mathbf{x}|\ |\mathbf{r}| \tan\theta = \mathbf{x} \cdot \mathbf{r} = |\mathbf{r}|.$$

The Diagonal in a Parallelogram



The diagonal within a parallelogram from lower left to upper right within an equilateral triangle has its largest length equal to the side of the equilateral triangle. The top of the parallelogram is x_0, and as the diagonal r slides down the side z, x_0 and r become equal to x.

Remember the algorithmic method uses the triangular numbers to come up with the area of a parallelogram? Let y^2 be a unit of area within the equilateral triangle, and



$x_n = \sqrt{(y^2)}$.

$(x_1 = x) \Rightarrow y^2$,

$(x_2 = 2x) \Rightarrow 3y^2$,

$(x_3 = 3x) \Rightarrow 5y^2$,

: : :

$(x_n = nx) \Rightarrow y^2(2n - 1)$, but then,

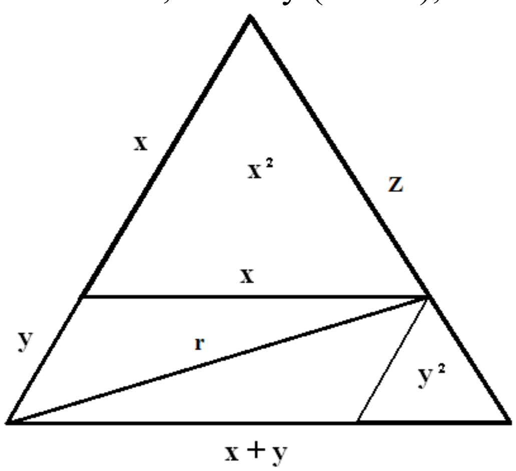
$(x_{n+1} = x(n + 1)) \Rightarrow y^2(2n + 1)$.

Now, $\Delta x^2 = y^2(2n + 1)$, but subtract one y^2 , and you get

$xy = \Delta x^2 - y^2$, so

$xy = y^2(2n + 1) - y^2$, and

$xy = 2ny^2$.



I have been saying that taking the triangular root of an area gives you a line. Then is it true that from

$$\Delta x^2 = y^2(2n + 1),\ x = y\sqrt{(2n + 1)},$$

giving the length of diagonal r as r approaches x? As diagonal r sweeps down the z axes, the triangular number nx approaches the triangular number x(n + 1) with the next triangular number as nx + x(n + 1) = 2nx + x = x(2n+1) which is the number of equilateral triangles within the parallelogram given by $xy = y^2(2n + 1) - y^2$ so that
r => 2n + 1as y^2 approaches zero.

Volume becomes squished into a line, so 2n + 1 is the greatest length r can attain, where n = 0, 1, 2, 3, … , t. In the slice Δx^2 of x^2 at the bottom of the triangle xyz that includes the parallelogram xy and the equilateral triangle y^2, there are (2n + 1) y^2's and (2n + 1) x's in r as r approaches x. In other words, r => Δx^2 as r => x, becoming two-dimensional. Δx^2 becomes very thin, becoming dx^2.

Let's start with the (2n + 1) y^2's. Take the first y^2 and take the triangular root of it. Take the next y^2 and do the same on down the line. For an ever expanding triangle, there are (2n + 1)y^2's along the bottom of the triangle. Adding up all the little volumes within the triangle, and taking the triangular root,

$x^2 = y_1^{\ 2} + 3y_2^{\ 2} + 5y_3^{\ 2} + \ldots + (2n + 1)y_n^{\ 2} \Rightarrow y_1 + y_2\sqrt{3} + y_3\sqrt{5} + \ldots + y\sqrt{(2n + 1)} = x$.

Looking at each slice singly,

$$\Delta x^2 = y_1^{\ 2} \Rightarrow y_1 = x$$

$$\Delta x^2 = 3y_2^{\,2} \Rightarrow y_2\sqrt{3} = x$$
$$\Delta x^2 = 5y_3^{\,2} \Rightarrow y_3\sqrt{5} = x$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\Delta x^2 = (2n + 1)y_n^{\,2} \Rightarrow y\sqrt{(2n + 1)} = x$$

This is not reasonable. Intuitively, the (2n + 1) should not be under the root sign. Within $y\sqrt{(2n + 1)}$, n would have to increase at an alarming rate for this expression to form all of the triangular numbers. After taking the triangular root of each y^2 in Δx^2 , all the y's are added up to create x as y^2 reduces to zero, that is, as r swings down to be parallel to x.
Therefore, $y_1^{\,2} + y_2^{\,2} + y_3^{\,2} + \ldots + (2n + 1)y_n^{\,2} \Rightarrow y(2n + 1) = x$, that is, (2n + 1) y's. (Pronounced "wise.")

[Notice that the amount of x's on top of the slice Δx^2 is nx, and on the bottom is nx + 1. Add these together and we get the number of equilateral triangles in the slice: nx + (nx + 1) = 2nx + x = x(2n+1). Since we are talking about unit x's and unit y's, x(2n + 1) = y(2n + 1). Example, 1 + 2 = 3, 2 + 3 = 5, 3 + 4 = 7, 4 + 5 = 9, etc. as n = 0, 1, 2, 3, … , t. This is called triangular numbers.]

There is a general principle here. It has to do with factoring. Take an expression V^2 and factor out an x^2 such that $Nx^2 = V^2$ for any expression N. Now take the triangular root of both sides so that $x\sqrt{N} = V$. That is the traditional way of doing it. My objection is that this doesn't work in a 60^o coordinate system. For any expression $Nx^2 = V^2$, taking the triangular root of both sides we have Nx = V. The N is never put under the root sign. It is independent from the operation of taking the triangular root. So having a volume V^2 , and taking its triangular root, making it a line V, we have a length Nx giving the number of unit x's. If it were $\sqrt{N}$ x's, it would be too complicated to fit in with a 60^o coordinate system in that $n \varepsilon N$ would have to increase at an alarming rate, skipping some slices of the triangle, to create all of the triangular numbers. But Nx provides the right order as in the example above.

Approaching an Algebra of the 60^o Coordinate System

We have some really important discoveries pertaining to the algebra of a 60^o coordinate system.

1. $x + y = z$ as the distance equation instead of $z = \sqrt{(x^2 + y^2)}$,
2. $x = \sqrt{(x^2)}$, showing that the triangular root of an area is a line,
3. A parallelogram is treated as a rectangle xy when its internal angles are 60^o and 120^o ,
4. $(x + y)^2 = x^2 + xy + y^2$ and not $x^2 + 2xy + y^2$ because a triangle is 1/2 a square, and
5. If x^2 is a factor of V^2 , $Nx^2 = V^2$, then $Nx = V$.

Note: If $y_n^{\,2} = y^2/(2n + 1)$ and $y_n^{\,2}[(n + 1)/(2n + 1) + n/(2n + 1)] = y^2 = \Delta x^2$ then let F be called a Fourier, so that $F = \Delta x^2 - y_n^{\,2}(n/(2n + 1))$
Note: all the triangles within the major triangle are counted using triangular numbers. This

leads to being an analog for harmonics of a string. This leads to the orthogonality of sin and cosine functions. This leads to the basis vectors of the 60° coordinate system.

Simplifying Mathematics

With these ideas in mind, we can build a foundation for simplifying mathematics. (I figured this out on my own without consulting a text book. Afterward, I remembered seeing the equation x = r cos θ. I had to prove to myself that this equation was true. But it is true for a 60° coordinate system as well as a 90° coordinate system. It is an invariant between these two coordinate systems. That is what I love about mathematics when I find something like this.)

The radius of a circle becomes r = ax + by so there is no need for the Pythagorean Theorem. You have to choose your coordinate system carefully to simplify math.

You can see from the above that the (2k + 1) is a triangular number. The Fourier Series seems to fit right into the 60° coordinate system.

Beginnings [edit]

> $$\varphi(y) = a_0 \cos\frac{\pi y}{2} + a_1 \cos 3\frac{\pi y}{2} + a_2 \cos 5\frac{\pi y}{2} + \cdots.$$
>
> Multiplying both sides by $\cos(2k+1)\frac{\pi y}{2}$, and then integrating from $y=-1$ to $y=+1$ yields:
>
> $$a_k = \int_{-1}^{1} \varphi(y)\cos(2k+1)\frac{\pi y}{2}\,dy.$$
>
> —Joseph Fourier, Mémoire sur la propagation de la chaleur dans les corps solides. (1807)[9][nb 3]

This immediately gives any coefficient a_k of the trigonometrical series for φ(*y*) for any function which has such an expansion. It works because if φ has such an expansion, then (under suitable convergence assumptions) the integral

$$\begin{aligned} a_k &= \int_{-1}^{1} \varphi(y)\cos(2k+1)\frac{\pi y}{2}\,dy \\ &= \int_{-1}^{1} \left(a \cos\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2} + a' \cos 3\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2} + \cdots\right)\,dy \end{aligned}$$

can be carried out term-by-term. But all terms involving $\cos(2j+1)\frac{\pi y}{2}\cos(2k+1)\frac{\pi y}{2}$ for $j \neq k$ vanish when integrated from −1 to 1, leaving only the *k*th term.

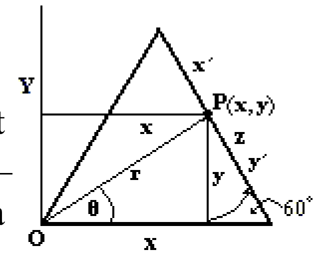
In these few lines, which are close to the modern formalism used in Fourier series, Fourier revolutionized both mathematics and physics. Although similar trigonometric series were previously used by Euler, d'Alembert, Daniel Bernoulli and Gauss, Fourier believed that such trigonometric series could represent any arbitrary function. In what sense that is actually true is a somewhat subtle issue and the attempts over many years to clarify this idea have led to important discoveries in the theories of convergence, function spaces, and harmonic analysis.

When Fourier submitted a later competition essay in 1811, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: *...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.*[citation needed]

from: http://en.wikipedia.org/wiki/Fourier_series

Conversion Between 90° and 60° Coordinates

Overlay a 60° coordinate system over a 90° coordinate system such that they both share the same origin at O and vector r. Choose any point P(x, y) at the end of r. Draw a line x from the Y–axis to P and another line y from the X–axis to P. Draw a line Z from the X–axis through P such that the angle between Z and X is 60° . Call y' the distance along Z from the X–axis to P. Assuming that Z is the right side of an equilateral triangle with a base along the X–axis, then let x' = z – y' along the length of z above P. The length of any side of this equilateral triangle is z = x' + y' by definition of an equilateral triangle. Since x' may be unknown, z can be found another way. A right triangle is created by y and y' having a base x" where x" = y' cos 60º. To obtain z simply add x + x". So, in order to convert between a 90° coordinate system and a 60° coordinate system, one has to know the x and y coordinates of P. Then y' can be calculated using y, x" can be calculated using y', and then z = x + x" and x' = z – y'. If the length of z is known beforehand, the calculations simplify, and z can be arbitrarily as long as P(x, y) is on z.



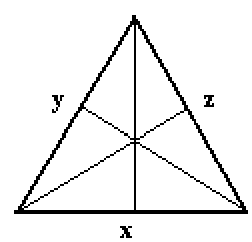
If z is known, then using the definition of the trigonometric function, $y = r \sin \theta$, it can be seen that $y' = y/\sin 60°$ and then $x' = z - y'$.

If z is not known, then

$$y' = y/\sin 60°,\ x'' = y' \cos 60°,\ x + x'' = z, \text{ and then } x' = z - y'.$$

Parametric Equations

Using the lines from the corners out to the opposite sides,
x = z/c, y = x/c, or z = y/c, then the length of these lines are:
$x = r \cos \theta$, (cos θ is the x coordinate in the unit triangle)
$y = r \sin \theta$, (sin θ is the y coordinate in the unit triangle) and
$z = r \sec \theta$, where sec θ is the z coordinate in the unit triangle.
These six equation constitute the parametric equations of a line.



Length of a Line

Adding these equations, we can come up with a generalized equation for r.
If $r = x/\cos \theta$ or $r = y/\sin \theta$, then adding these two equations,

$2r = x/\cos \theta + y/\sin \theta$, and

$r = \frac{1}{2}(x/\cos \theta + y/\sin \theta)$.

Generalizing,

$2r = [(x_1 - x_0) / \cos \theta] + [(y_1 - y_0) / \sin \theta]$, such that

$r = \frac{1}{2} ([(x_1 - x_0) / \cos \theta] + [(y_1 - y_0) / \sin \theta])$, and for three dimensions,

$r = 1/3 ([(x_1 - x_0) / \cos \theta] + [(y_1 - y_0) / \sin \theta] + [(z_1 - z_0) / \sec \theta])$.

Shape of a Line

The shape of a line is given by using the equations for constant lines crossing a single point P(x, y,z),

$x = c$,

$y = c$,

$z = c$,

and adding these equations to get some curve

$x + y + z - 3c = 0$, or generalizing,

$ax + by + cz - k = 0$.

P(x, y, z) is a point of intersection of the three above lines on any curve that can be drawn within the equilateral triangle as a combination of x's y's and z's to produce the lines necessary to that point. This is the same principle as a pen on a graph machine.

The Distance Formula

Instead of, as in a 90° coordinate system where the distance formula is

$d = \sqrt{[(x_1 - x_0)^2 + (y_1 - y_0)^2]}$,

the distance formula for the side of an equilateral triangle or radius of its arc is

$d = x + y$.

For a rotated line inside an equilateral triangle, or for that part of the radius of an arc that is inside the triangle,

$d = \xi / \text{trig}_{\xi}\ \theta$ where $0^{\circ} < \theta \leq 60^{\circ}$, or from the parametric equations,

$d = \frac{1}{2} (x / \cos\theta + y / \sin\theta)$ which is an average of two lines.

Note: using all three coordinates, $d = 1/3 (x / \cos\theta + y / \sin\theta + z / \sec\theta)$.

Now with the knowledge we have in $x + y = z$, we know that z is a side of an equilateral triangle and z^2 is the area of that triangle. Now within that area, the triangle is divided into an x^2, a y^2, and xy, a parallelogram. So $z^2 = x^2 + xy + y^2$, and $z = \sqrt{(x^2 + xy + y^2)}$. It's not the Pythagorean Theorem, but it does give an accurate description of a length of a line withing a 60° coordinate system.

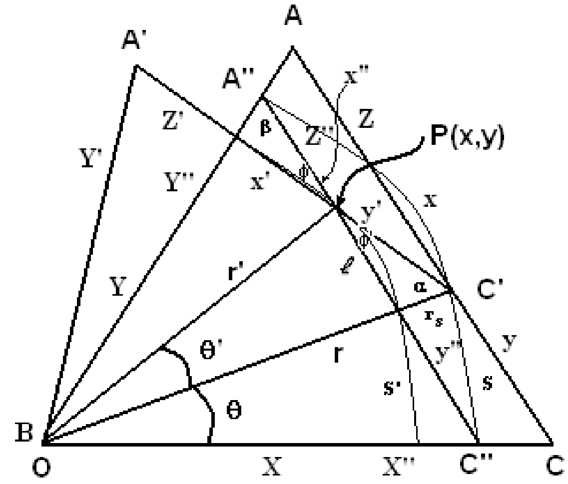
The Length of r

Any line r has a length of x + y. Within an equilateral triangle ABC with sides X, Y, and Z, draw a line **r** from O to the opposite side Z. Line **r** is the base of another equilateral triangle A'BC' and is a radius drawing out an arc s. Swing a copy of **r** down to X so that **r** coincides with X. From the endpoint of **r**, draw a line l parallel to Z up to Y. Line l completes another equilateral triangle A"BC". The triangle A'BC' therefor is only the triangle A"BC" rotated at an angle of θ.

Z' = Z", **r** = X", and Y' = Y" by definition. Draw a line **r'** from O to Z' where triangles

A'BC' and A"BC" intersect at point P(x,y). Because Z' crosses Z", producing angles ϕ and ϕ', the angles ϕ and ϕ' are equal.

Radius r' divides Z' into x' and y' and Z" into x" and y".



All angles of equilateral triangles are equal, so α and β are equal by definition.

Since Z' = Z", y' / x' = x"/ y".

The distance between l and Z is $\mathbf{r_s}$. The distance between the two arcs s and s' of which both extend from X" to Y" is also $\mathbf{r_s}$.

Angle α is made up of sides y' and $\mathbf{r_s}$ and angle β is made up of sides x" and $\mathbf{r_s}$.

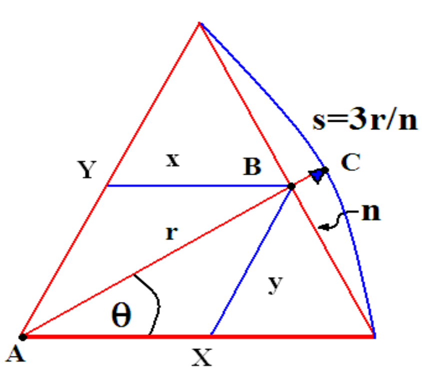
Because y' / x' = x"/ y" and $\alpha = \beta$ and $\phi = \phi'$, then the triangles A'A"P(x,y) and C 'C " P(x,y) are congruent.

The triangles A'A"P(x,y) and C 'C " P(x,y) are both shortened by sides $\mathbf{r_s}$. We can call the side opposite ϕ, $\mathbf{y_s}$.

Because $\alpha = \beta$ and $\phi = \phi'$ and $\mathbf{y_s} = \mathbf{r_s}$, then y' = x".
If y' = x" and Z' = Z", then x' = y".
By definition, Z" = X" = Z' = **r**.
Therefore, **r** = x' + y' = x" + y".

Generally speaking then, any line segment within an equilateral triangle, no matter at what angle it lays, be it l or r, can be defined by z = x + y.

Angles

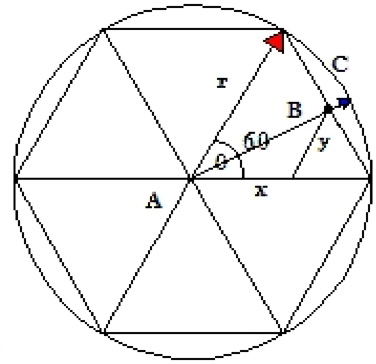


(

Starting with an equiangular, equilateral triangle, let there be a vector extending from the lower left corner through the opposite side such that the vector's length is the same as that of any side of the triangle.

Referring to the figure to the left, at the point **C**, the length of the vector $\mathbf{r = x + y}$. But at the point **B**, the full length of $\mathbf{r}$ = ½ x / cos θ + y / sin θ). (In the figure, n = y.)

The figure to the right shows a hexagon. It represents the



cycle of the circle based upon the chords of the hexagon instead of the ratio π of the circumference to the diameter. Using a unit triangle where each side is one unit, and therefore, each chord is one unit, the circumference of the circle is divided into 6 parts. Let one rotation of **r**, beginning and ending at the x–axis, stand for one cycle. Each chord represents $1/6^{th}$ of a cycle. Then it is logical to keep that division of 6 and divide each chord into 6 equal parts or $1/36^{th}$ of the hexagon. Then there would be 36 divisions of the circle. If there are then 10 divisions between each $1/36^{th}$ mark, there will be 360 divisions around the hexagon. Extending those divisions to the enclosing circle, the circle then receives 360 divisions. The angle θ can be represented by these divisions being projected onto the circle, which would be 360^o. Divisions on the hexagon would be 360 radians. Using radians, larger divisions would be in terms of n/36, and smaller divisions, n/360.

The trigonometric functions are defined as follows:
$\sin \theta = y/z$
$\cos \theta = x/z$
$\tan \theta = y/x$
$\cot \theta = x/y$,
and $\theta = \tan^{-1} y/x$ where y will always be in divisions of n/36 or n/360.

A trigonometric table within the unit triangle of the hexagon using radian measure s based upon $1/36^{th}$ of a cycle and $1/360^{th}$ of a cycle:

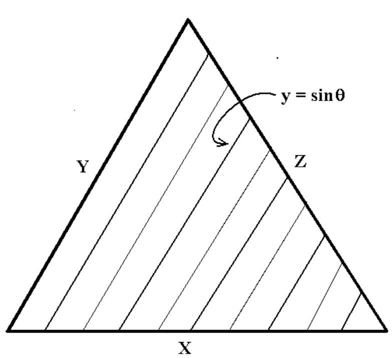
s	***Sin*** *s*	***Cos*** *s*	***Tan*** *s*
0	0	1	0
1/360, 1/36	1/360, 1/36	359/360, 35/36	1/359, 1/35
1/180, 1/18	1/180, 1/18	179/180, 17/18	1/179, 1/17
1/120, 1/12	1/120, 1/12	119/120, 11/12	1/119, 1/11
1/90, 1/9	1/90, 1/9	89/90, 8/9	1/89, 1/8
1/72, 5/36	1/72, 5/36	71/72, 31/36	1/71, 5/31
1/60, 1/6	1/60, 1/6	59/60, 5/6	1/59, 1/5
7/360, 7/36	7/360, 7/36	353/360, 29/36	7/353, 7/29
1/45, 2/9	1/45, 2/9	44/45, 7/9	1/44, 2/7
1/40, 1/4	1/40, 1/4	39/40, 3/4	1/39, 1/3
1/36, 5/18	1/36, 5/18	35/36, 13/18	1/35, 5/13
1/30, 1/3	1/30, 1/3	29/30, 2/3	1/29, 1/2
1/20, 1/2	1/20, 1/2	19/20, 1/2	1/19, 1
1/10, 1	1/10, 1	9/10, 0	1/9, +1

Note that in the Tangent column above all the denominators are one less than the denominators in the s column, and also in the Tangent column, the numerator added to the denominator equals the denominator in the s column. Also, look at the placement of the nines and multiples of nine.

36 * 1/360 segments = 1/10 (sin 6^{o})
18 * 1/360 segments = 1/20 (sin 3^{o})
12 * 1/360 segments = 1/30 (sin 2^{o}), 5 * 1/30 = 1/6, and then 6 * 1/6 = 1
10 * 1/360 segments = 1/36, and then 6 * 1/36 segments = 1/6. Then 6 * 1/6 segments = 1 unit
09 * 1/360 segments = 1/40
08 * 1/360 segments = 1/45
07 does not work. It does not divide evenly into 360.
06 * 1/360 segments = 1/60 (sine 1^{o}), 10 * 1/60 = 1/6, then 6 * 1/6 = 1 unit.
05 * 1/360 segments = 1/72, 6 * 1/72 = 1/12 (sin 5^{o}), 2 * 1/12 = 1/6, then 6 * 1/6 = 1 unit.
04 * 1/360 segments = 1/90
03 * 1/360 segments = 1/120, 2 * 1/120 = 1/60, 10 * 1/60 = 1/6, then 6 * 1/6 = 1 unit.
02 * 1/360 segments = 1/180, 3 * 1/180 = 1/60, 10 * 1/60 = 1/6, then 6 * 1/6 = 1 unit.
(Notice that the 6 in (sin 6^{o}) times 6 is 36, the 3 in (sin 3^{o}) times 6 is 18, etc. showing a relationship of 6 to degree measure and the numbers multiplying 1/360.)

Setting one segment of a chord or radian to $1/360^{th}$, only 2, 3, 5, 6, 10 or 12 segments produce a rational multiple of the 6 sections of a hexagon. (That is, under the count of 36.)

The divisors of 360 include all the digits except 7, but only 6 and 60 divide 360 in a symmetry that the other digits don't. This is because $36 = 6^2$.
360/2 = 180, 360/20 = 18; 360/3 = 120, 360/30 = 12; 360/4 = 90, 360/40 = 9; 360/5 = 72, 360/50 = 7.2; 360/6 = 60, 360/60 = 6; 360/8 = 45, 360/80 = 4.5; 360/9 = 40, 360/90 = 4.
The only divisor here that creates symmetry is 6 because x/6 = 60 and x/60 = 6 where the 6 and the 60 are interchangeable and there are no other digits you can do this with. Therefore, it seems more natural to divide the circumference of a circle into 6 sections, or multiples of 6.



There are 12 * 30^{o} segments in a circle and 5 * 72^{o} segments in a circle. Every 30^{o} is divided into 5 * 6^{o}. If the circle is divided into 12^{o} segments, every 5^{th} segment is 60^{o} . Half of each 12^{o} segment would be 6^{o} . So dividing the circle into 6 degree segments, you get the numbers 5, 6, 12, 36, 60, 72, and 360, getting multiples of 2, 3, and 5.

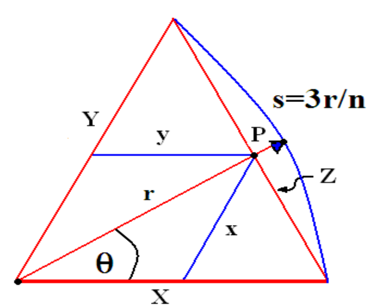
Next, look at the trigonometric functions using degrees. This trigonometric table is based on a unit triangle within a unit hexagon. The degrees of a circle are projected onto the Z axis of the equilateral triangle. Each coordinate pair of the projected points then correspond to a degree. What is listed below is either x/z or y/z where z is the radius of the circle encompassing a hexagon. Therefore, the trigonometric functions are based upon the chords of the circle instead of the arcs of the circle.

θ	***Sin θ***	***Cos θ***	***Tan θ***
0°	0.0000000	1.000000	0.0000000
5°	0.1015625	0.906250	0.1120690
10°	0.1875000	0.812500	$0.\underline{230769}$...
15°	0.2734375	0.734375	0.3723404
20°	0.3515625	0.656250	0.5357143
25°	0.4218750	0.5781250	$0.\underline{729}7297$
30°	0.5000000	0.5000000	1
35°	0.5781250	0.4218750	$1.\underline{370}370$
40°	0.656250	0.3515625	1.8666...
45°	0.734375	0.2734375	2.685714
50°	0.812500	0.1875000	4.3333...
55°	0.906250	0.1015625	8.923077
60°	1.000000	0.0000	∞

Sec θ = Sin θ or Cos θ because a secant is the chord underneath the arc. In other words, Sec θ = x + y which is the z coordinate. (Errors may be due to computer calculations.) (Underlined segments are to be repeated.)

Directed Line Segments or Vectors

If r is treated as a vector, then $r = (x_1 - x_0) + (y_1 - y_0)$. This agrees with the definition of a vector, that is, v = i(cos θ) + j(sin θ) where v is a unit vector. Multiplying both sides of the equation by a constant c, where u = cv, u = ci(cos θ) + cj(sin θ). If x = ci (cos θ) and y = cj (sin θ), then u = x + y . (It must be remembered that the trigonometric functions used here are based upon the 60° coordinate system, so the numbers will be different.)



If $\theta > 60^\circ$, then l is being rotated into the next sextant, and the calculations of length are similar, and is similar for each of the six sextants.

Another line, radius r, can be considered having the same length of any side of an equilateral triangle. When it is rotated between 0° to 60° then r intersects Z at any point P. The coordinates of P are the components x and y of r such that x + y = Z, and x + y = r since r = X = Y = Z .

To add r_1 and r_2,
$r_1 = x_1 + y_1$, and
$r_2 = x_2 + y_2$, so
$r_1 + r_2 = x_1 + y_1 + x_2 + y_2 = (x_1 + x_2) + (y_1 + y_2)$ which adds up to a larger triangle.
$r_1 - r_2 = (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2)$ which is a decrease in the triangle.

Looking at the previous figure, you can see that $z_0 = x_0 + y_0$ but x_0 and y_0 are components of z_1. Seeing that
$z_1 = x_0 + (x_1 - x_0) + y_0 + (y_1 - y_0)$ then,
$r = z_1 - z_0 = x_0 + (x_1 - x_0) - x_0 + y_0 + (y_1 - y_0) - y_0$ and,
$r = z_1 - z_0 = (x_1 - x_0) + (y_1 - y_0)$ so,
$r = z_1 - z_0$.
When r is rotated outside the triangle
$d = r_1 + r_2 = (x_1 + x_2) + (y_1 + y_2)$ or,
$d = r_1 - r_2 = (x_1 - x_2) + (y_1 - y_2)$ or,
$d = r = z_1 - z_0$.

If the vector $r = x + y$, and
$\sin \theta = y/r$ and
$\cos \theta = x/r$, then
$\cos \theta + \sin \theta = (x + y)/r$. Therefore,
$\cos \theta + \sin \theta = 1$.
Solving for $(x + y)$, we have, $(x + y) = r(\cos \theta + \sin \theta)$. Therefore,
$z = r(\cos \theta + \sin \theta)$.

Let $s = (x + y)$, so any vector $s = r(\cos \theta + \sin \theta)$. Another way of writing s is re^{θ} where $e^{\theta} = (\cos \theta + \sin \theta)$. So the vector $s = re^{\theta}$.

$(\cos \theta + \sin \theta)$ can be represented by the ordered pair (x, y). Therefore, at each of the six axes of the hexagon, at $r = n$, $\cos \theta$ and $\sin \theta$ are replaced by:

for the x–axis, (1, 0),
y–axis, (0, 1),
z–axis,+(0, 0),
– x–axis, (–1, 0),
– y–axis, (0, –1), and
– z–axis, –(0, 0).

If a third number were included into the ordered pair to make an ordered triplet, then a 4th –axis would be represented thusly: (x, y, z) and would be inside a tetrahedran.

The difference between a vector and a scalar

A vector **r** can be extended only in one way, and that is to be multiplied by a scalar a, thus, a**r**. Although two vectors can be added to make a longer vector, the original vectors are not extended. The only way a scalar can be extended is to have something added to it, thus, x + ω. In discussing the conic sections, I will deal with the extensions of both vectors and scalars.

The Conic Sections

Coordinate System and Lines

Take any finite line L having ends A and B. Divide it into any two sections x and y. The point of division is P(x, y), the sections of the line being the coordinates of the point. Have the length of L be z such that z = x + y. P now describes any point on the line L with x and y the coordinates.

Let L be expanded into a plane such that L is one side of an equilateral triangle ABC where C is a point opposite L = AB = z_0. Let x as well as y be expanded such that x as well as y is the side of an equilateral triangle congruent with triangle ABC. Let AC = y_0 and CB = x_0. The base of the triangle Ay_0P, the triangle expanded from x on L, is parallel to x_0 and is equal to x and is called the x coordinate. The side of the triangle PBx_0 parallel to y_0 is equal to y and is equal to the y coordinate.

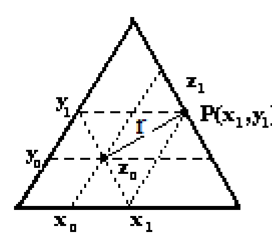
Let there be parallel lines z_0, z_1, z_2, … z_n descending down to point C in the triangle ABC where z_0 = L. Then draw a line l from z_{n-1} to z_n dividing z_{n-1} and z_n into x_{n-1} and y_{n-1} and x_n and y_n respectively. The endpoints of l are (x_{n-1} , y_{n-1}) on z_{n-1} and (x_n , y_n) on z_n.

The length r of line l can be thought of as a vector. Remember that r = ($x_{n-1} + x_n$) + ($y_{n-1} + y_n$) = x + y, with (x, y) as the endpoint. The norm of r, |r| = $x_n + y_n$.

The Circle

If r is held constant and and the endpoint of r, (x_n , y_n), remains attached to all z_n descending down towards point C, then r becomes the radius of a circle. If z_n passes through the circle, there is another point (x_n' , y_n') opposite (x_n , y_n) on z_n such that (x_n' , y_n') is also the endpoint of r. (x_n' , y_n')is the conjugate of (x, y).

For an arbitrary line segment r in the middle of the triangle, we draw a line z_0 through the lower end of r and z_1 at the other end of r. r divides z_0 into ($x_1 - x_0$) and ($y_1 - y_0$). ($y_1 - y_0$) extended straight across to z_1 and ($x_1 - x_0$) extended up to z_1 creates a parallelogram with sides ($x_1 - x_0$) on top and bottom and ($y_1 - y_0$) on each side with r as the diagonal.



The length of r can be treated as a vector. The sides ($x_1 - x_0$) and ($y_1 - y_0$) of the parallelogram are the components of r and can be added such that

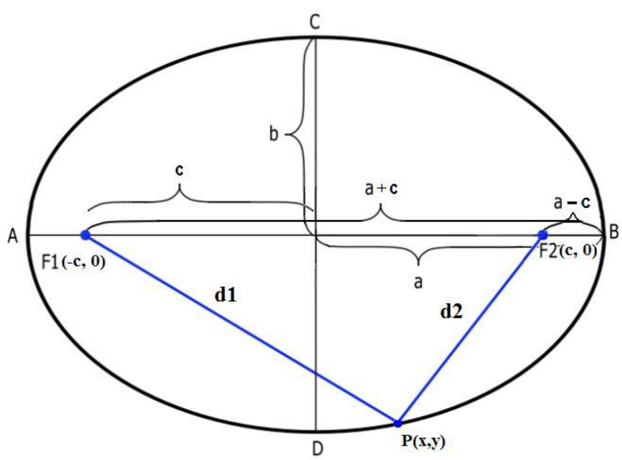
$$r = (x_1 - x_0) + (y_1 - y_0) = x + y .$$

To generalize this somewhat to include negative numbers in x and y, in other words, using the whole plane, which is a hexagon, let $r = |x_1 - x_0| + |y_1 - y_0|$ as the equation of a circle.

Vector addition is as simple as $|x_1 - x_0| + |y_1 - y_0| = (x, y) \Rightarrow z = x + y$ where x and y are coordinates.
Note: $x_1 + x_0 = x$ and $y_1 + y_0 = y$.

This is a geometric addition and describes a circle as r rotates and remains constant. The points (x, y) are on the curve of the circle. The circle is described geometrically as $r = x + y$.

The Ellipse and Proving the Distance Formula



A circle is defined as an ellipse where the distance c from the focus to the center is equal to 0. The distance from the center out to a point B on the ellipse is a. The distance from a focus out to a point P is d. Since there are two foci, $F_1(-c, 0)$ and $F_2(c, 0)$, there are two distances, d_1 and d_2, out to the same point P(x, y). The sum of the distances $d_1 = |F_1, P|$ and $d_2 = |F_2, P|$ is equal to some constant. When P is at B on the image to the left, then $d_1 + d_2$ is equal to that same constant. Therefore, $(a + c) + (a - c) = 2a$ is that constant.

But when P is not at B, $d_1 = (x - c) + y$, and $d_2 = (x + c) + y$, so $(x - c) + y + (x + c) + y = 2a$. Since $x - c$ and $x + c$ are always changing relative to the same y, let $x - c = x_1$ and $x + c = x_2$. Then the equation for an ellipse becomes

$$x_1 + x_2 + 2y = 2a. \text{ This becomes}$$
$$y = a - (x_1 + x_2)/2.$$

The general equation becomes $2x + 2y = 2a$ or $x + y = a$. As c becomes zero, a becomes radius r, so the equation for a circle becomes $r = x + y$.

In order to find the point P on the curve in terms of x and y, we use three vectors, (x, y) on the curve, (x_0, y_0), the left focus, and (x_1, y_1), the right focus, where the origin of all three vectors is at the corner formed by the x and y axes. That makes $d_1 = (x_0, y_0) - (x, y)$ and $d_2 = (x_1, y_1) - (x, y)$. So

$$(k, 0) = [(x_0, y_0) - (x, y)] + [(x_1, y_1) - (x, y)], \text{ and}$$
$$(k, 0) = (x_0, y_0) - 2(x, y) + (x_1, y_1). \text{ So}$$
$$(x, y) = [(x_0, y_0) + (x_1, y_1) - (k, 0)] / 2$$

Letting $y = 0$,
$$(x, 0) = [(x_0, y_0) + (x_1, y_1) - (k, 0)] / 2$$
$$x = [(x_0 + x_1) + (y_0 + y_1) - k] / 2$$

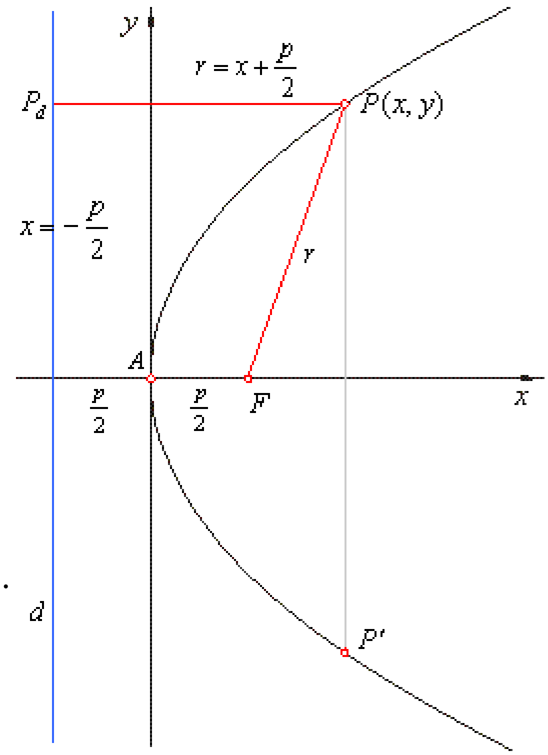
From the above paragraph, the equation for the ellipse is

$$y = [k - (x_1 + x_2)] / 2, \text{ where } k = 2a.$$

It could also be interpreted as $y = [(x_1 + x_2) - k] / 2$.

It is seen that the general equation is $z = x + y$.

The Parabola



The apex A of a parabola is equidistant from a line parallel to the y axis (or in some cases, the x axis), called the directrix D, and a point F, called the focal point of the curve. Traditionally, the distance from the directrix and the focal point is called p. Now A, as it travels along the parabolic curve, is the point P(x, y). This point is always equidistant from the directrix and the focal point. That's what makes it parabolic. The smallest distances (D, A) and (A, F) are p/2. Let the distance from F to P be r_0, and the distance from D to P be r_1.

Therefore, $r_0 = r_1$.

A lies on the y axis. The equation of this line is x = 0. The equation of the directrix is $x = -p/2$. r_1 then is the distance from Pd(– p/2, y) on the directrix to P(x, y) on the curve. r_0 is the distance from P(x, y) to the focal point F(p/2, 0). When y = 0, that is, when P = A, so that also x = 0, $r_1 = -p/2$ and $r_0 = p/2$. But when P starts rising or lowering from A, P has to be added to the equation. So $r_1 = (-p/2 + y) - (x + y)$ and $r_0 = (p/2 - 0) - (x + y)$.

Now we have the equation of the parabola as

$$(p/2 - 0) - (x + y) = (-p/2 + y) - (x + y),$$
$$(p/2 - x) - (y + 0) = -(p/2 + x) - (y - y).$$
$$(x - p/2) - y = -(x + p/2), \text{ so}$$
$$y = (x - p/2) + (x + p/2).$$

The vector representation of the parabola goes in this manner.

Let $F = (x_0, y_0)$, $P = (x, y)$, and $P_d = (x_1, y_1)$, so that using vectors, we have

$$(x_0, y_0) - (x, y) = (x, y) - (x_1, y_1)$$

This reduces to

$$2(x + y) = -(x_1 + y_1) - (y_0 + x_0)$$
$$2x = -x_1 - y_1 - y_0 - x_0 - 2y, \text{ then}$$
$$x = [-(x_0 + x_1) - (y_0 + y_1) - 2y] / 2$$
$$x = -[(x_0 + x_1) + (y_0 + y_1) + 2y] / 2$$

It appears that the general equation for the parabola is also

$$z = x + y.$$

The Hyperbola

The difference between the ellipse and the hyperbola is that with the ellipse, r_0 and

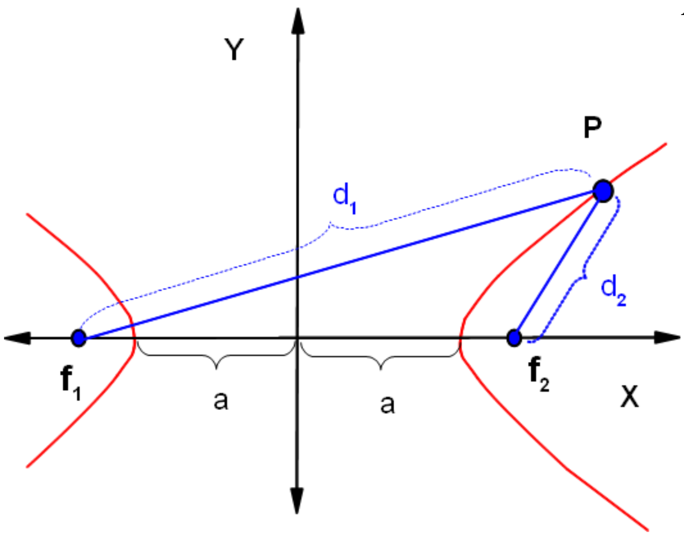
r_1 are added. The hyperbola is created with a difference of r_0 and r_1 . Therefore, the equations for the hyperbola is

$$y = a - (x_1 - x_2)/2,$$

and using vectors,

$$x = [(x_0 + x_1) - (y_0 + y_1) - k] / 2, \text{ where } k = 2a.$$

Again, the general formula is $z = x + y$.



Comparison of Equations

The ellipse:

$$x = [(x_0 + x_1) + (y_0 + y_1) - k] / 2$$

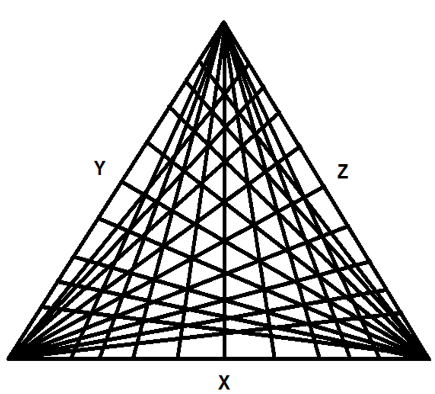
The hyperbola:

$$x = [(x_0 + x_1) - (y_0 + y_1) - k] / 2$$

The parabola:

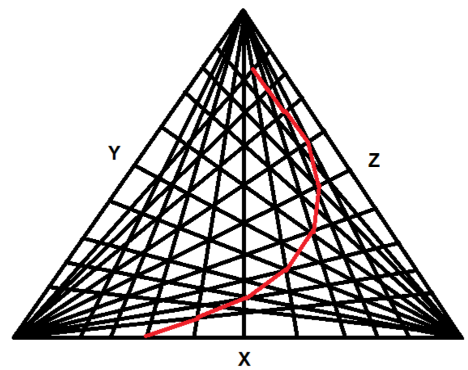
$$x = - [(x_0 + x_1) + (y_0 + y_1) + 2y] / 2$$

These three equations are linear and deal with straight lines. The constant in these cases are parallel to the z axis or one side of the equilateral triangle. The traditional equations, I would say, deal with the area of the equilateral triangle. Changing the above equations to volumes is as simple as triangling them. These lines are the triangular roots of the volumes.

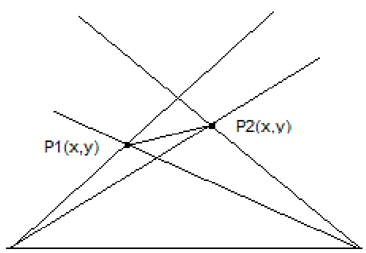


Towards a Theory of Conics

For all lines $\xi = q\xi'$ within an equilateral triangle, space is bent around an isometric space representing a three dimensional 90° space. This defines a linear space for three dimensions. Using these lines as a grid, wherever any two sets of lines z = by, x = cz, y = ax, and z' = by', x' = cz', and y' = ax', intersect, two points P_1(x, y, z) and P_2(x', y', z') are created. Draw a straight line l going from P_1 to P_2 such that the line goes from one corner of a parallelogram to another. Draw another diagonal line within the next parallelogram, then onto the next, etc. Drawing a straight line through a curved space, which the linear space within a triangle is, the line appears to follow the curve of the space. The finer the grid, the smoother the curve. The curve created in this manner is non–linear, but because it is made up of linear segments, it is discrete. Any curve in Nature is made up of discrete linear segments. There is no true continuous line. Everything in nature comes in little packets.

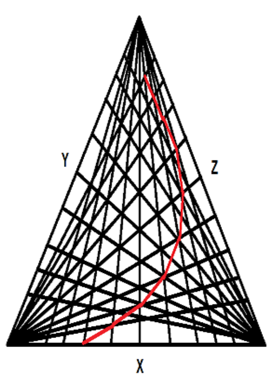


Any conic section, any curve, is not continuous. It is made up of small discrete lines. A compromise with mathematicians who argue about continuity cannot be made. Even if you use a 90^o coordinate system, space must be curved using small discrete segments for curved lines. Any curve can be created using a linear space.

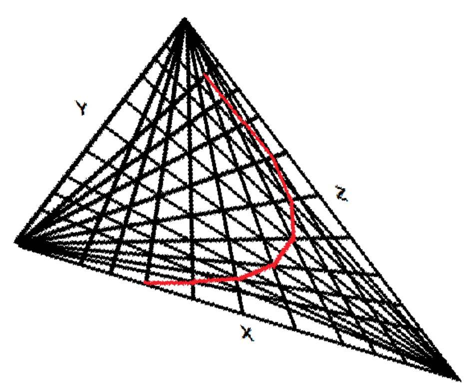


First, using all points made from the intersection of equations ξ = qξ', find a set of line segments that form a parallelogram. Draw a line from P_2(x, y, z) and P_1(x, y, z) which can be found by solving four simultaneous equations of the form ξ = qξ'. Then include the next set of four equations to solve them and the last set as the next four equations and so on until you have all the points in the curve, then connect all the points.

From ξ = qξ', we get ξ – qξ' = 0. If we sum all of Δξ – Δqξ', we get the curve $\sum_{yo}^{yn}$(Δξ – Δqξ').



If you skew the triangle, you have a differently shaped curve. Since each triangle has a one–to–one correspondence to the 60^o triangle by scale, we are still within the 60^o coordinate system. Any curve can be demonstrated by curving the space it is drawn in.



The Rotating Line Segment

It is true that z = x + y, and for z = 1, x = cos θ, y = sin θ, so that

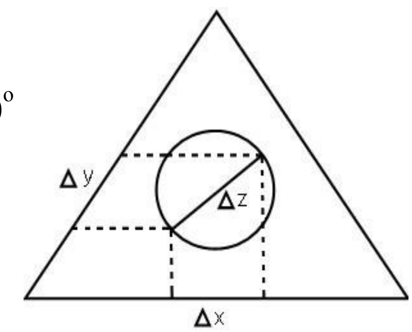
$1 = \cos\theta + \sin\theta$, where $0^\circ < \theta \leq 60^\circ$.

Multiply both sides by Δz to get

$\Delta z = \Delta z \cos\theta + \Delta z \sin\theta$, which is true for $0^\circ < \theta \leq 60^\circ$.

Let there be a Δx and a Δy such that $\Delta z \cos\theta = \Delta x$ and $\Delta z \sin\theta = \Delta y$ where Δx is a variable line segment on the x–axis and Δy is a variable line segment on the y–axis and θ is the angle between Δz and Δx.

Therefore, $\Delta z = \Delta x + \Delta y$.



Proof: [It is sufficient to show that $\Delta z = \Delta z \cos\theta + \Delta z \sin\theta$.]

Let $\Delta z = ½ \Delta z + ½ \Delta z$. Therefore, $\Delta z = \Delta z \cos 30^\circ + \Delta z \sin 30^\circ$

Generalizing, if it is true for 30°, it is also true for $0^\circ < \theta \leq 60^\circ$.

Therefore, $\Delta z = \Delta z \cos\theta + \Delta z \sin\theta$.

Remember, that $\cos\theta + \sin\theta$ can be expressed as e^θ. Therefore, another way to express Δz is $\Delta z\, e^\theta$**.** Since $\theta = \omega\tau$, then $\Delta z = \Delta z\, e^{\omega\tau}$ and is a line segment rotating at the angular velocity of ω.

Theorem:

The rotating line segment Δz rotates about its midpoint.

Definitions: Vertical means lying at 60°. Horizontal means lying at 0° or 180°.

Proof: [It is sufficient to show that $\Delta z \neq \Delta x + \Delta y$.]

Δz is a constant, as $\Delta z = \Delta x + \Delta y$. If Δz is horizontal, then $\Delta z = \Delta x$, and $\Delta y = 0$. The endpoints therefore of Δz lie on the curve of a circle whose diameter is Δx. If Δz is vertical, then $\Delta z = \Delta y$, $\Delta x = 0$, and the endpoints of Δz lie on the curve of a circle whose diameter is Δy. If Δz is neither vertical nor horizontal, then $\Delta z = \Delta x + \Delta y$. Say that Δz rotates through any point other than its center, esp. near either endpoint, then each endpoint moves along a different circle and both Δx and Δy are split. Therefore, Δz would not equal $\Delta x + \Delta y$.

Circular Measurements

The circumference of a circle $C = 2\pi r$.

Arc length $s = r\,\theta$.

To change from degrees to radians, $\theta = \pi/n$, or divisions of π.

Therefore, in radians, arc length $s = r\,\pi/n$.

Now if we base $\boldsymbol{\pi}$ on the perimeter of the unit hexagon instead of on the circumference of the unit circle, and since the circumference of a circle is divided by a hexagon into six arcs, let $\pi = 3$. Where π would be, there are 3 chords.

Substituting 3 for π, arc length s = 3 r/n.

Let $\pi = C/2r$. The circumference of the circle now becomes $C = 2 \cdot 3r = 6r$. It isn't how many times the radius fits around the circumference, but 6 chords of the circumference times the radius outwards, the circumference expanding.

Arc measure verses Linear Measure

A circular hexagon is likened unto a hexagon that is blown up like a balloon. The sides of the hexagon are bent so as to produce a circle with the same radius that exists from the center of the hexagon to one of its corners. Let this radius be one unit. This circular hexagon inscribes the regular hexagon. Draw another circle with six equal arcs around the circumference so that the radius equals the same length as one of the arcs.

Let sub h designate the first hexagon and the sub c designate the second circle.

The circumference $c_h = 6\ r_h$. An arc $a_h = 3r/n$

Let radius $r_h = 1$. Therefore arc $a_h = 1$ and $c_h = 6$.

Since $c_c = 6\ r_c$, and $r_c = 3r/n$, then $c_c = 18r/n$ where n is some division of 360 in 60 increments.

Having the radius the same length of the arc shows a different sized circle than the circle as a blown up hexagon.

The area of a spherical triangle $\Delta = r^2 [(a + b + c) - \pi]$ where a + b + c = e, called the spherical excess (which is actually 6^o). If e = 0, Δ is the area of a planar triangle. If each side of one of the 20 spherical triangles on the surface of the spherical icosahedron is equal, then a = b = c = 72^o or 1.2564 radians. If r = 1, the area of each one of these spherical triangles is $\Delta = (3 \text{ x } 1.2564) - \pi = 0.627607$. Multiply that by 20 and you get the surface area of 12.5521 square radians. That converts to 14 triangular radians and to 720^o (which is the sum of all the angles in a tetrahedron). But divide 0.627607 by (√3)/4 to convert it to equilateral triangles and multiply that by the 20 spherical triangles of the spherical icosahedron and you get 29 = 4 x **7** 1/4 (4 times the area of one great circle). The surface area of a unit sphere is $4\pi r^2$. With r = 1, $4\pi = 12.5521$, but 12.5521/(√3)/4 = 29. So the surface area of a sphere is $29r^2$ in spherical equilateral triangular units.

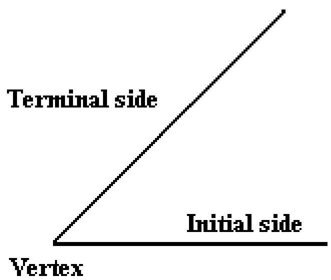
There are new expressions for each of the three dimensions in equations of nature's way of measuring that don't use π or the Pythagorean theorem.

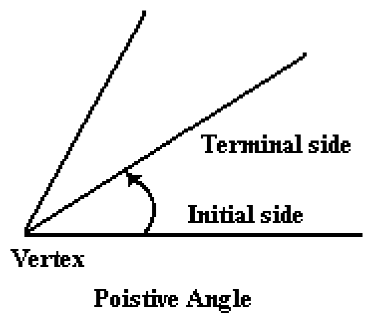
Old Formula	New Formula	Units
$V_{sphere} = 4/3\ \pi r^3$	$5\ r^3$	tetrahedrons
	$\mathbf{7\ ^1/_4\ r^2}$	triangles

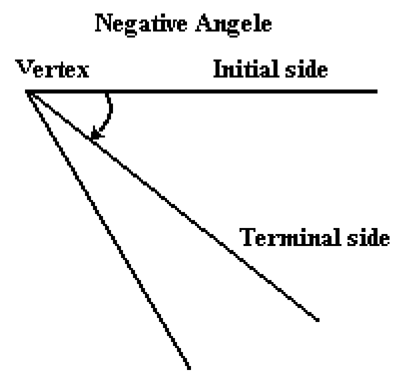
$A_{circle} = \pi r^2$		
$A_{sphere} = 4 \pi r^2$	**29** r^2	spherical triangles
$C_{circle} = 2 \pi r$	6 r	60^o arcs
$S_{arc} = r \pi/n$	3 r/n	60^o arcs

(Note also that $4/3 \pi r^3$ where $\pi = 3$ equals 4, but $4 \times (9/8)^2 = 5$)

A Review of Trigonometry







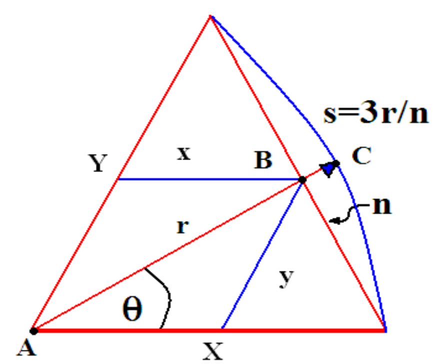
Starting with the angle, it has an initial side, a terminal side and a vertex or the point of the angle. The standard position of the angle, called positive, is a counterclockwise rotation. The negative angle then has a clockwise rotation. An angle may be generated by making more than one revolution, the terminal side passing the initial side once or more than once. And depending upon the direction of rotation, a negative angle remains a negative angle, and a positive angle remains a positive angle.

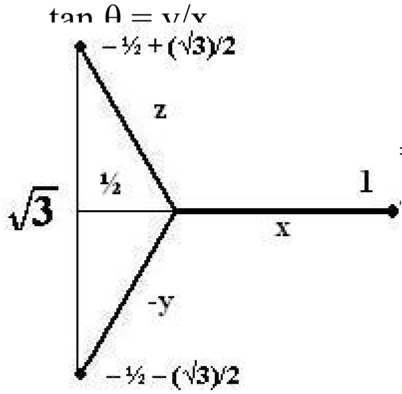
The trigonometric functions can be defined as follows:

$\sin \theta = y/(x+y)$,

$\cos \theta = x/(x+y)$,

$\tan \theta = y/x$





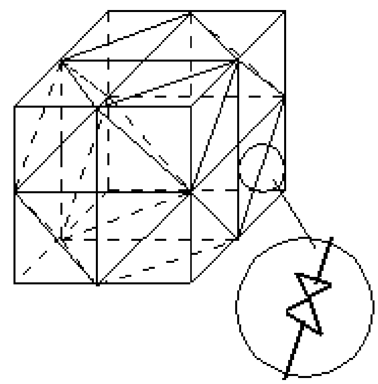
z.

of n/36 or n/360.

Traditionally, trigonometric functions have been based upon the right triangle. The tetrahedral function $z^3 = +/- 1$ forms an interface between 90° and 60° coordinate systems. Based upon the right triangle, Sin 60° has been defined as (√3)/2, which is the volume of a unit equilateral triangle, and cos 60° as ½. The distance between the imaginary roots of $z^3 = 1$ is √3 which is also defined as tan 60°. The diagonal of a square with sides of √2 is sec 60° which is 2, the reciprocal of cos 60°. So, we have

sin 60° = (√3)/2,
cos 60° = **½,**
tan 60° = √3 and
sec 60° = 2.

The cos 60° = ½ is only one half the base of an equilateral triangle. The hypotenuse of the right triangle is unity and is another side of that triangle. Allowing the hypotenuse be defined as the y–coordinate and 2 cos 60° = ½ be defined as the x–coordinate of the 60° coordinate system, we have been led from our 90° coordinate system into our 60° coordinate system. This is true for every 30°/60°/90° triangle.



Each diagonal is actually two vectors pointing to each other.

Take a cube of which each side is divided into four squares where each side is (√2)/2, making that a cube of √2. Drawing one inch diagonals to each of the four squares on each face of the cube so that they form another square which is turned 45° from the square face, each of these diagonals can be connected to form 4 hexagons interlaced within the cube. Each of the diagonals is ½ sec 60° = |x| = |y| = |z| = 1, and this shows the 45° angle between the 90° and 60° coordinate systems.

Imaginary numbers also form an interface between 90° and 60° coordinate systems. Using the side s of an equilateral triangle, an imaginary number is of type, +/– s/2 +/– h, where h is the height of the triangle. For a unit equilateral triangle, the imaginary number would be +/– ½ +/– (√3)/2. Since the height of the triangle would have as many divisions n as the side s, a general imaginary number would be +/– s/2n +/– h/n. But since the height of a unit equilateral triangle is (√3)/2, the imaginary number would be +/– s/2n +/– (√3)/2n. The distance between any two of – s/2n – (√3)/2n, + s/2n – (√3)/2n, – s/2n +(√3)/2n, + s/2n + (√3)/2n would be |(√3)/n|. In a 90° coordinate system, sin 60° = (√3)/2n, and cos 60° = s/2n. The imaginary number is actually +/– cos 60° +/– sin 60°. That would be true whether it is in the 90° coordinate system or the 60° coordinate system. Therefore it is an interface.

To turn this to a 60° coordinate system, n would have to be equal to 2/(√3), and s = 0. The result would be sin 60° = 1 and cos 60° = 0. On the other hand, the sin 60° of the 60° coordinate system is equal to twice the cos 60° of the 90° coordinate system. This is due to the fact that, taking half of a unit equilateral triangle, which is a right triangle, with the angle opposite the height of the triangle equal to 60°, the base is one half the hypotenuse. The hypotenuse is the sine of the angle in the 60° coordinate system, whereas the base is the cosine of the angel in the 60° coordinate system. The height is not taken into consideration.

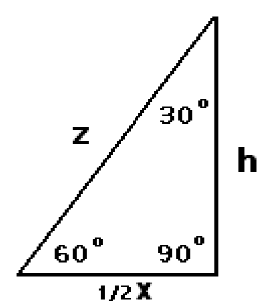
The sec of 60° is 2 which is the length of two connected sides of the unit hexagon. The tan of 60° is √3 which is the length of a line connecting the two ends of the two connected sides of the unit hexagon. With this information, a table of the trigonometric functions can be produced.

A trigonometric table based on a single triangle within a unit hexagon using degree measure q:
Even though degree measure is used, the result is the same as radian measure because it is based upon chords of the circle and not the circle itself.

θ	*Sin θ*	*Cos θ*	*Tan θ*	*Cot θ*
0°	0	1	0	∞
10°	1/6	5/6	(√3)/12, 1/5	5
15°	1/4	3/4	(√3)/8, 1/3	3
20°	1/3	2/3	(√3)/6, ½	2
30°	½	(√3)/2, ½	(√3)/4, 1	1
40°	2/3	1/3	(√3)/3, 2	1/2
45°	1/(√2), 3/4	1/(√2), 1/4	1, 3	1/3
50°	5/6	1/6	5(√3)/12, 5	1/5
60°	(√3)/2, 1	½, 0	(√3), ∞	0

(Sec θ = Sin θ or Cos θ because a secant is the chord underneath the arc. In other words, Sec θ = x + y. (On the 30°, 45°, and 60°, as well as the Tan θ column, I have included measures from the 90° coordinate system.)

Also included in trigonometric measurement is the functions of secant, cosecant, and cotangent which are the reciprocals of cosine, sine, and tangent, respectively.

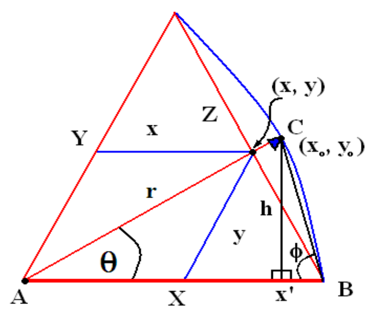


I include here a table of imaginary numbers obtained from the right triangle.

Z	*h*	*½ x*	*X*	*I–numbers*
1	(√3)/2	1/2	1	+/– ½ +/– (√3)/2
2	√3	1	2	+/– 1 +/– √3
3	3(√3)/2	1 1/2	3	+/– 1½ +/– 3(√3)/2

4	2(√3)	2	4	+/– 2 +/– 2√3
5	5(√3)/2	2 1/2	5	+/– 2½ +/– 5(√3)/2
6	3(√3)	3	6	+/– 3 +/– 3√3
7	7(√3)/2	3 1/2	7	+/– 3½ +/– 7(√3)/2
8	4(√3)	4	8	+/– 4 +/– 4√3
9	9(√3)/2	4 1/2	9	+/–4½ +/– 9(√3)/2

The radius r of a circle is the hypotenuse of a right triangle. The perpendicular leg h extending down from the point (x_o, y_o) on the circle is the sine of the opposite angle, whereas the base of the triangle is the cosine of the same angle, the angle θ which the radius makes with the x–axis. Let this circle enclose a hexagon such that the sides of the hexagon are the chords of the circle. The point (x, y) where the radius r intersects the hexagon is the start of the 60^o coordinate system. A line y extending down from the point (x, y) to intersect the x–axis at 60^o is the y coordinate. From that point x back to the origin is the x coordinate. These two lines x and y added together give the same length as the radius r. x' is the base of the right triangle.



r – x' is the distance between the two points (x_o, y_o) on the arc and (x, y) on the chord.

Why?

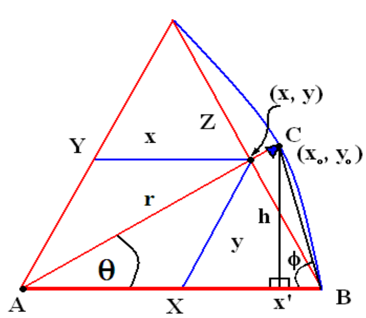
First, because X = r. That's a given.

Second, the right triangles Cx'B and C(x, y)B share the same side CB, the chord of the arc between C and B.

Third, the angle between r and CB and the angle between X and CB are equal,and

Fourth, angle Bx'C = angle C(x, y)B, they both being right angles, and since there are two angles in the two right triangles that are equal plus the fact that they share one side, the two triangles are equal.

Therefore, (x, y)C = x'B. A vertical dropped down from (x, y) gives you the imaginary number. y = h.



Why?

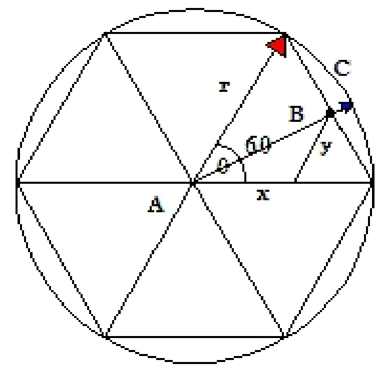
First, the two right triangles are equal.

Second, the two bases of the right triangles are equal.

Third, the two right triangles share the same side CB.

Let the two equal bases be b. Let the side CD be a. Then $h = \sqrt{(a^2 + b^2)}$. But (x, y)B is also equal to $\sqrt{(a^2 + b^2)}$, and y = (x, y)B because (x, y)B is part of an equilateral triangle where y is one of the sides, so, h = y.

Now if h = sin θ , then y is also equal to sin θ. cos θ is merely found by subtracting y from r.



The point (x, y) is on the secant z, and not on the circle c. If the length of the secant approaches zero, then we might say that the point (x, y) coincides with a point on c, but this mathematics is more concerned with points on the hexagon.

Remember that $x = r \cos \theta$ and $y = r \sin \theta$.

For a point extending from the side of a smaller hexagon within a larger hexagon to the side of that enclosing hexagon, and the two hexagons share a common center, then the outer point $(x_o + r \cos\theta, y_o + r \sin\theta)$ is an extension of the inner point (x_o, y_o).

Chapter Five
A Geometric Algebra
(inspired by David Orlin Hestenes)

Multiplying Vectors

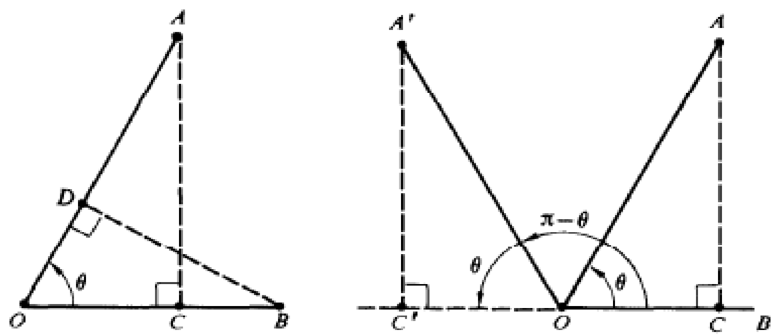


Fig. 4.1. Perpendicular projection.

The perpendicular projection of OA on the line OB is the line segment OC. The adjective "perpendicular" expresses the stipulation that the line AC be perpendicular to OC. If θ is the radian measure of the angle AOB and if $a = OA$ is the length of OA, then, one can write

$$OC = a\cos\theta.$$

Likewise, if $b = OB$, then

$$OD = b\cos\theta.$$

In the second diagram, to distinguish the projection of OA' on OB from the projection of OA on the same line where $OA' = a = OA$, it is convenient to regard OC' and OC as the line segments with the same magnitude but opposite orientation. This can be expressed by writing

$$OC' = a\cos(\pi - \theta) = -a\cos\theta = -OC.$$

This scalar quantity clearly depends on the orientation of OB, because OB determines the line from which the angle is measured.

By taking orientation into account, we go slightly beyond the Greek idea of perpendicular projection.

(taken from New Foundations for Classical Mechanics by David Hestenes, p. 17)

Algebraic systems fail to indicate the difference between scalars and vectors. This difference is not reflected in the rules for addition, but in the different geometric interpretations. Descates gave rules for "multiplying" line segments in which the direction didn't matter, and the result was a dilation or projection. We can take a different path and use rules for multiplication of line segments in which the direction does matter, that is, use a construction for the multiplication of two vectors. The construction that is familiar is the perpendicular projection of one line segment onto another. If only the relative direction of the line segments to be multiplied is needed, the result can be multiplied by the magnitude of either of them.

The <u>Inner Product</u> , also called the Dot Product, of two directed line segments **a** and **b,** written **a** • **b** is defined to be the oriented line segment obtained by dilating the projection of **a** onto **b** by the magnitude of **b**. The resulting line segment is a scalar.
The definition of **a** • **b** implies the relationship of **a** and **b** to the angle θ by

$$a \bullet b = |a||b| \cos\theta.$$

In Euclidean space, the inner product is always positive,

$$a^2 = a \bullet a > 0 \qquad \forall a \neq 0.$$

We define the cosine of the angle between the two vectors a and b as

$$a \bullet b \,/\, |a||b| = \cos\theta.$$

(In non–Euclidean spaces we cannot do this. We can however introduce an orthogonal frame and compute the dot product as in Minkowski space–time as $a_\mu b^\mu$ or $\eta_{\mu\upsilon} a^\mu b^\upsilon$, where $\eta_{\mu\upsilon}$ is the metric tensor.)

The reason that the resultant line segment is a scalar is because we are talking about relative direction in the definition of **a • b** and not the direction of either **a** or **b**. This way, the angle between **a** and **b** remains constant and produces an important symmetry property of

$$a \bullet b = b \bullet a. \quad (5.1)$$

The projection of **a** onto **b** dilated by |**b**| gives the same result as **b** projected onto **a** and dilated by |**a**|.

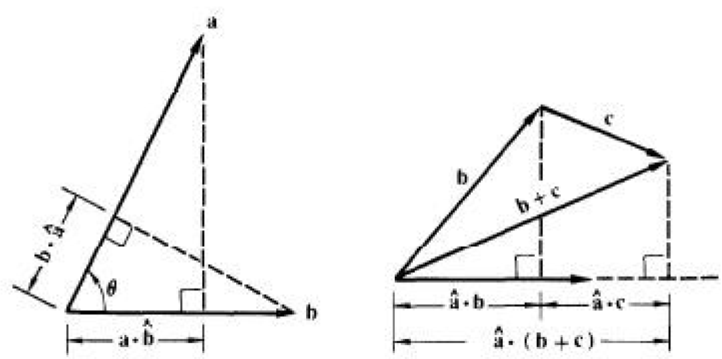


Fig. 4.2. Symmetry of the Scalar Product.

The perpendicular projections of **a** on **b**, and **b** on **a** give respectively

$$\mathbf{a}\cdot\hat{\mathbf{b}} = |\mathbf{a}|\cos\theta,$$
and
$$\mathbf{b}\cdot\hat{\mathbf{a}} = |\mathbf{b}|\cos\theta,$$

which after dilation results in the symmetrical form

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{b}|\,\mathbf{a}\cdot\hat{\mathbf{b}} = |\mathbf{a}|\,\mathbf{b}\cdot\hat{\mathbf{a}} = \mathbf{b}\cdot\mathbf{a}.$$

Fig. 4.3. Distributive rule.

The inner product is distributive with respect to vector addition. Projection gives

$$\hat{\mathbf{a}}\cdot(\mathbf{b}+\mathbf{c}) = \hat{\mathbf{a}}\cdot\mathbf{b} + \hat{\mathbf{a}}\cdot\mathbf{c}.$$

Multiply this by |**a**|, and use the distributive rule for scalars. Finally use the relation (4.3) and $|\mathbf{a}|\,\hat{\mathbf{a}} = \mathbf{a}$ to get the general distributive (4·4).

(taken from New Foundations for Classical Mechanics by David Hestenes, p. 19)

Two other algebraic properties can be deduced for the definition of the inner product. Its relation to scalar multiplication of vectors is expressed by

$$(\lambda a) \bullet b = \lambda(a \bullet b) = a \bullet (\lambda b) \text{ where } \lambda \text{ is positive, negative or zero.} \quad (5.2)$$

Its relation to vector addition can be expressed by the distributive rule:

$$a \bullet (b + c) = a \bullet b + a \bullet c. \quad (5.3)$$

The magnitude of a vector is related to the inner product by

$$a \bullet a = |\,\mathbf{a}\,|^2 \geq \mathbf{0}. \quad (5.4)$$

Of course, $a \bullet a = \mathbf{0}$ if and only if $a = 0$.

The inner product can be used to compare angles and the lengths of line segments.

Important theorems of geometry and trigonometry can be proved by the methods of vector algebra. This can be realized by using a simple vector equation for a triangle, viz.

$$c = a + b\ .$$

By using the inner product and squaring, using the distributive rule, one gets an equation relating the sides of a triangle to its inner angles:

$$\begin{aligned} c \bullet c &= (a + b) \bullet (a + b) \\ &= a \bullet (a + b) + b \bullet (a + b) \\ &= (a \bullet a) + (b \bullet b) + (a \bullet b) + (b \bullet a)\ , \text{ then,} \end{aligned}$$

$| c |^2 = | a |^2 + | b |^2 + 2(a \bullet b)$ which can be expressed in terms of scalars as

$$c^2 = a^2 + b^2 + 2ab \cos C.$$

This formula is called the law of cosines in trigonometry. If C reduces to zero, we have the Pythagorean Theorem. Therefore, the inner product is a link between the 60^o coordinate system and the 90^o coordinate system.

Similarly, from $a = c - b$ and squaring, one gets the the law of sines,

$$\sin A\ / a = \sin B / b = \sin C / c\ .$$

The Inner Product can be fully defined as a rule relating scalars to vectors having the properties of equations 5.1 through 5.4.The results of geometric and trigonometric theorems can be obtained easily from the algebra. For example, the fact that lines a and b are perpendicular can be written as $a \bullet b = 0$. The algebra becomes a useful language for describing the real world.

Note: The Cross Product, exists only in 3–D such that $a \times b$ is perpendicular to the plane defined by a and b. It has a magnitude of $|a||b| \sin \theta$. Also, a, b, and $a \times b$ form a right–handed set. Introducing a right–handed orthonormal frame $\{e_i\}$, we have

$$e_1 \times e_2 = e_3, \text{ etc.}$$

or generalizing, using index notation,

$$e_i \times e_j = e_{ijk}\, e_k.$$

Now if we expand the vectors in terms of components $a = a_i e_i$ and $b = b_i e_i$, we have,

$$\begin{aligned} a \times b &= (a_i e_i) \times (b_j e_j) \\ &= a_i b_j (e_i \times e_j) \\ &= (\varepsilon_{ijk} a_i b_j)\, e_k. \end{aligned}$$

But the geometric definition is in terms of frames. One aim of Geometric Algebra is to avoid introducing frames as much as possible.

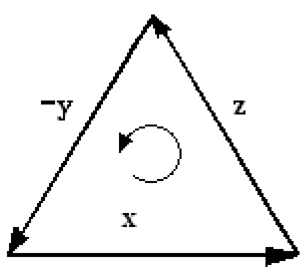
The Outer Product

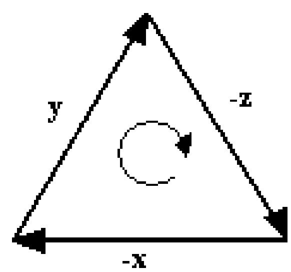
The failing of the cross product is that it exists only in 3 dimensions. It cannot exist in 2 dimensions. With an outer product, we can encode a plane geometrically without relying upon a vector perpendicular to it. We define an outer product as the area swept out by a and b. This is denoted as a /\ b, called 'a wedge b'. The plane has an area of $|a||b| \sin \theta$ which is defined to be the magnitude of a /\ b. There have been many systems such as tensor algebra, matrix algebra and spinor algebra designed to express this geometrical idea. All of these other systems can be expressed using this outer product.

> In the words of David Hestenes,
> The principle that the product of two vectors ought to describe their relative directions presided over the definition of the inner product. But the inner product falls short of a complete fulfillment of that principle, because it fails to express the fundamental geometrical fact that two non–parallel lines determine a parallelogram. (New Foundations for Classical Mechanics by David Hestenes, p. 21)

But we are thinking about the equilateral triangle and the geometric property of area as the geometrical product of two sides of the triangle expressed as the third side.

The result of the outer product is neither a scalar nor a vector. It is a new mathematical entity. It is the notion of an oriented plane segment or area. It is called a bivector. It can be visualized as an equilateral triangle with one vector sweeping back into the other. (The vector showing the direction of sweep is always the third vector opposite the angle created by the other two vectors.) Changing the order of the sweep reverses the orientation of the plane.





– y /\ x = z
a /\ b = c
x is swept towards – y, b into a
or x is swept across z

or

– z /\ – x = y
b /\ a = – c
– x is swept towards – z, a into b
and – x is swept across y

Recalling the tetrahedral roots of one, there were two triangles, each rotating in opposite directions. These are defined as outer products (similar to the cross product x **x** y in 3 dimensions, but applied to 2 dimensions), such that x /\ z = – y = 1, or – z /\ – x = y = 1. (These triangles are related to the equations a + b = c and a – c = – b.)

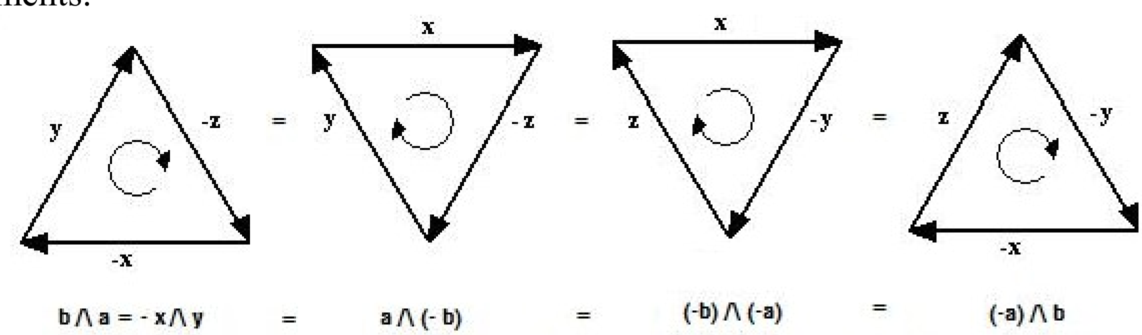
Generalizing, a /\ b = c and b /\ a = – c. Let c always point upward. Therefore, c going counterclockwise is z, and c going clockwise is y. This will be the standard. a /\ b is directed counterclockwise and b /\ a is directed clockwise. Then a is always swept into b clockwise, and b is always swept into a counterclockwise. Let the outer product be defined as the area of a unit directed equilateral triangle, so if a = b = 1, a /\ b = 1, and b /\ a = – 1 . In Nature's Way of Measuring, a unit bivector has the area of 1.

The bivector is not a set of points but is a directional relation of that set of points, specifying the the plane the points are in. As a vector is a directional relation of points on a line, the bivector characterizes the directional relation of points within an area of a plane.

Because $b \wedge a = - a \wedge b = - c$, the outer product is anticommutative. The relation between vector orientation and bivector orientation is fixed by the rule:

$$b \wedge a = a \wedge (-b) = (-b) \wedge (-a) = (-a) \wedge b. \quad (5.4)$$

If left crosses over to right, it changes sign, but if right crosses over to left, it does not change sign. This comes from the correspondence of vectors and bivectors with line segments and plane segments.



This can be expressed with scalars as

$$y - x = - z, \qquad x + y = - z, \qquad x - y = z, \qquad - x - y = z$$

This is because if y and z have both been rotated 180^{o}, then x remains the same to keep the same triangular orientation, but if y and z are then rotated 60^{o}, then x is rotated 180^{o}. A rotation of 180^{o} does not change the sign of a the third number, but a rotation of 60^{o} does.

The magnitude of $a \wedge b$ is just the area of an equiangular equilateral triangle. Therefore,

$$|B| = |a \wedge b| = |b \wedge a| = |a|\,|b| \sin\theta, \quad (5.5)$$

where θ is the angle between a and b and $0^{o} \leq \theta \leq 60^{o}$. If that is the case, then θ is found within a hexagon which is divided up into six 60^{o} angles. Therefore, this definition handles any angle.

Scalar multiplication for bivectors is the same as for vectors. For any bivectors A and B and scalar λ , then

$B = \lambda A \quad (5.6)$

means that the magnitude of A is dilated by λ , that is,

$|B| = \lambda\, |A|$

where the direction of A is the same as B unless λ is negative. This last stipulation can be expressed by multiplication by numbers 1 and – 1:

$B = B, (-1)\, B = - B. \quad (5.7)$

Bivectors which are scalar multiples of one another are said to be codirectional.

Scalar multiplications of vectors and bivectors are related by

$\lambda(a \wedge b) = (\lambda a) \wedge b = a \wedge (\lambda b)$. (5.8) (an either/or situation: either a is multiplied by λ or b is multiplied by λ.)

For $\lambda = -1$, this equation is the same as equation (5.4). For positive λ, equation 5.8 shows that dilation of one side of the triangle is dilated the same amount as the other side.

By equation 5.5, if $\sin\theta = 0$ then $|a \wedge b| = 0$ for non–zero a and b. This is the algebraic way of saying that a and b are collinear. It follows that $|a \wedge b| = 0$ if and only if $a \wedge b = 0$. (This follows the principle of vectors that if the magnitude of a vector is zero, then the vector is zero.) Therefore, the outer product $a \wedge b$ of non–zero vectors is zero if and only if they are collinear.

Assuming that

$a \wedge b = 0$ and $b = \lambda a$, then

$a \wedge a = 0.$

Using the anticommutative rule,

$a \wedge a = -a \wedge a$

for only zero is equal to its own negative. The vector a is collinear with itself.

From Area to Volume

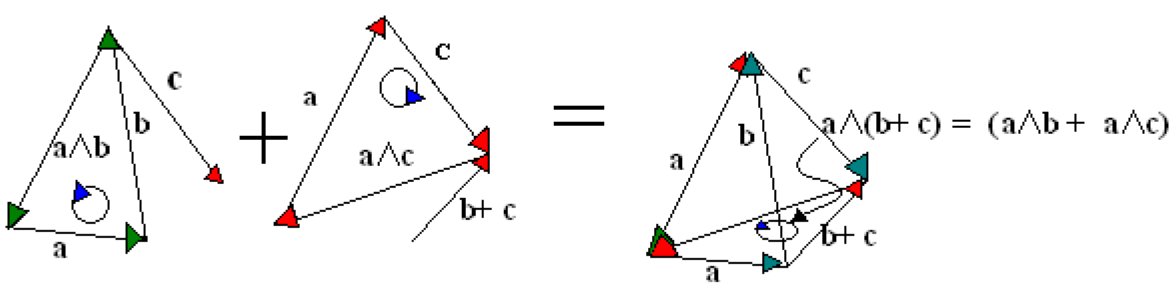
Outer multiplications and addition have a relationship in the distributive rule:

$$a \wedge (b + c) = a \wedge b + a \wedge c.$$

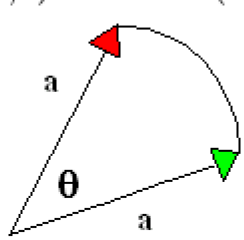
This relates addition of bivectors on the right to the addition of vectors on the left. The algebraic properties of bivectors are completely determined by the properties of vectors. The addition of two bivectors is a bivector and this addition is associative.

But what about $(a \wedge b) + (c \wedge d) = (a + c) \wedge (b + d)$? Is that true?

In 3–d the addition of bivectors is easily visualized. The plane of a bivector can be shown within a vortex or a twist of three vectors in three dimensions. Taking the three vectors of a vortex and combining them to three vectors of the opposite twist, a tetrahedron is formed showing the distributive property of bivectors.



(a, b, c) vortex (b+c, a/\c, a) vortex [a, b+c, a/\(b+c)] vortex



The a of a∧b and the a of a∧c are the same a, only a has been displaced by an angle θ. a is the radius of a circle or the radius of a sphere.

The inner and outer products are compliments of each other. They are both measures of relative direction. If a relation is too difficult to obtain using one, then it can be found with the other. Whereas the equation $a \cdot b = 0$ provides a simple expression for two perpendicular lines, $a \wedge b = 0$ provides a simple expression for two parallel lines.

Perpendicular lines > $a \cdot b = 0$
Parallel lines > $a \wedge b = 0$

Take the outer product, for example, of $a + b = c$ successively with vectors a, b, and c. This produces the equations

$$a \wedge b = a \wedge c$$
$$b \wedge a = b \wedge c$$
$$c \wedge a = c \wedge b$$

which can be expressed by

$$a \wedge c = a \wedge b = c \wedge b.$$

Expressed in magnitudes,

$$|a \wedge c| = |a \wedge b| = |c \wedge b|,$$

and using the definition of the outer product,

$$|a||c|\sin\theta = |a||b|\sin\theta = |c||b|\sin\theta,$$

using the scalar labels A, B, C for the angles of a triangle, and dividing by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

All the formulas for plane and spherical trigonometry can be derived using inner and outer products.

Properties of a bivector:

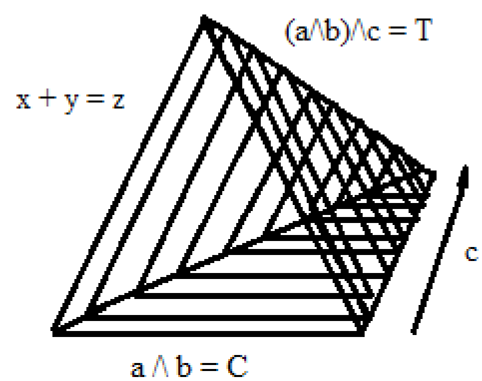
1. The outer product of two vectors is antisymmetric,
$$a \wedge b = -\, b \wedge a.$$
This follows from the geometric definition.
2. The outer product is distributive. That is,
$$a \wedge (b + c) = a \wedge b + a \wedge c.$$
3. Bivectors form a linear space the same as vectors do.

Bivectors were originally based upon directed parallelograms. Now, there is really no unique dependence on a and b. If $a' = a + \lambda b$ we still have $a' \wedge b = a \wedge b$. Let $\lambda b = c$. Therefore, we have $(a + c) \wedge b = a \wedge b + c \wedge b$ where $c \wedge b = \lambda b \wedge b = 0$, since λb and b are parallel. Thus, it is sometimes better to replace the directed parallelogram with a directed circuit, and the equilateral triangle is the simplest directed circuit.

Three Dimensions

The outer product can be generalized. Just as a triangular plane segment is swept out by a vector or line segment, a spacial segment of a tetrahedron is swept out by a plane segment. A bivector $a \wedge b$ moves at a distance and direction symbolized by the vector c producing a tetrahedron.

The algebra of Hamiltonian quaternions contains 4 elements {1, i, j, k}, but only three of these specify a vector. This can be generalized by defining the 1 as a fourth vector. {1, i, j, k} is generalized as {a, b, c, d}. This fourth dimension d can be interpreted as time, so the Hamiltonian can be demonstrated with a tetrahedron T, a trivector. We write the outer product of a bivector $a \wedge b$ with a vector c as



$$(a \wedge b) \wedge c = T.$$

As for bivectors, trivectors obey the associative rule:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c).$$

This rule can be determined from another rule: $(a \wedge b) = -(b \wedge a)$ such that

$$(b \wedge a) \wedge c = (-a \wedge b) \wedge c = -T .$$

Thus, the orientation of a trivector can be reversed by reversing the orientation of only one of its components. This makes it possible to rearrange the vectors to get

$$(a \wedge b) \wedge c = (b \wedge c) \wedge a = (c \wedge a) \wedge b,$$

which means that $(a \wedge b)$ sweeping along c, $(b \wedge c)$ sweeping along a, and $(c \wedge a)$ sweeping along b, all results in the same tetrahedron. But if

$$(a \wedge b) \wedge c = a \wedge b \wedge c = 0,$$

then c lies in the same plane as a and b, and a tetrahedron is not produced.

Also, as bivectors are anticommutative, so are trivectors:

$c \wedge b \wedge a = -\, a \wedge b \wedge c.$

Without such algebraic apparatus, any geometrical idea of relative orientation would be difficult to express. Adding more dimensions does not add any new insights into the relation between algebra and geometry. The displacement of a trivector $a \wedge b \wedge c$ along a fourth vector d does not produce a fourth–dimensional space segment analogous to a three dimensional tetrahedron, especially since the tetrahedron is enough to demonstrate a four dimensional manifold.

So $(a \wedge b \wedge c) \wedge d = a \wedge b \wedge c \wedge d = 0.$

The Geometric Product

The symmetric inner product $a \cdot b$ and the antisymmetric outer product $a \wedge b$ are combined in the geometric product, called a multivector,

$$ab = a \cdot b + a \wedge b.$$

It can also be thought of as composed of real and imaginary parts. The inner product is the real part.

Since $a \cdot b = b \cdot a$, and $a \wedge b = -\, b \wedge a$, by the symmetry/antisymmetry use of the terms,

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b.$$

It follows that by taking the sun and difference of the equations for ab and ba that

$$a \cdot b = \tfrac{1}{2}(ab + ba) \quad \text{and} \quad a \wedge b = \tfrac{1}{2}(ab - ba).$$

We can thus form other products in terms of the geometric product.

$$ab = \tfrac{1}{2}[(ab + ba) + (ab - ba)].$$

Remember that the area of a parallelogram is the determinant of two vectors or $[ad - bc]$. Half of that determinant is the area of a triangle. $a \wedge b$ is the area of a trinagle, so if $a \wedge b = \tfrac{1}{2}(ab - ba)$, then the difference of two multivectors $(ab - ba)$ is a determinant.

Properties of the Geometric Product:

1. General elements of a Geometric Algebra are called multivectors and are usually written in upper case, (A, B, C, …). These form a linear space in which scalars can be added to bivectors, and vectors, etc.
2. The geometric product is associative:
$$A(BC) = (AB)C = ABC$$
3. The geometric product is distributive:
$$A(B+C) = AB + AC$$
(Matrix multiplication is a good thing to keep in mind.)
4. The triangle of any vector is a scalar.

The proof of this last property is to prove that the inner product of any two vectors is a scalar.

Let $c = a + b$, and therefore, $c^2 = (a + b)^2$. Expanding,

$$(a + b)^2 = (a + b)(a + b)$$
$$= a^2 + b^2 + ab + ba$$

It follows that

$$ab + ba = (a + b)^2 - a^2 - b^2$$

In geometric algebra, $ab = C$ has the solution $b = a^{-1} C$. Neither the dot product nor the cross product are capable of this inversion on their own.

Geometric Algebra in 2–d

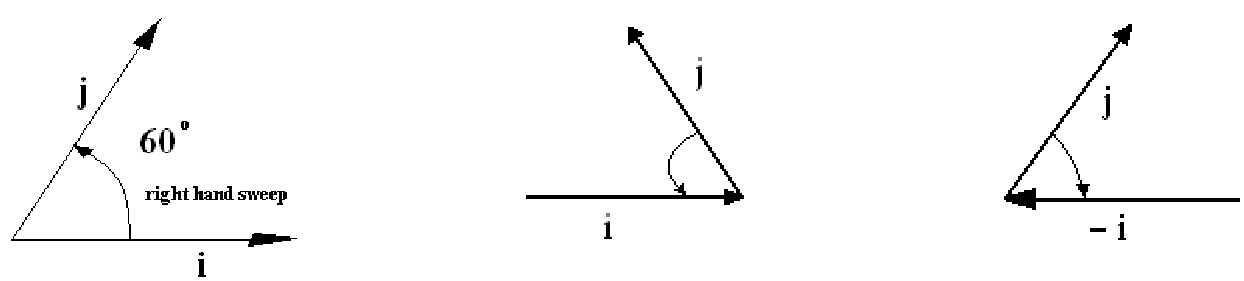
Consider a plane which is spanned by 2 orthonormal vectors i and j. These basis vectors

satisfy $i^2 = j^2 = 1$, $i \bullet j = 0$, and $|i| = |j|$.

The next entity present in a 2 dimensional algebra is the bivector $i \wedge j$. By convention,

bivectors are right–handed, so that i sweeps onto j in a right–handed sense when viewed from above. But if we keep the convention of j always being vertical, j sweeps into i either way.

classical arrangement the geometric product $i \wedge j$ j is always vertical



The other convention is that the head of one vector connects to the end point of the other vector.

Let $\boldsymbol{I} = i \wedge j = i\,j$. When $\boldsymbol{I} = i \wedge j$, we call $\boldsymbol{I}$ a pseudoscalar. The full algebra is spanned by

1	{ i , j}	$i \wedge j$
1 scalar	2 unit vectors	1 bivector

We denote this algebra by G_2. The law of multiplication for G_2 is that the geometric product

$$i\,j = i \cdot j + i \wedge j = i \wedge j\,.$$

That is, for orthogonal vectors, the geometric product is a pure bivector. Because of the anticommutativity property of bivectors, $i\,j = i \wedge j = j \wedge -i = -(j\,i)$. In Geometric Algebra, orthonormal vectors anticommute.

We can now form products from the right and from the left. Multiplying a vector by a bivector from the left, we rotate the vector clockwise 60^o. Let i be multiplied by ij on the left.

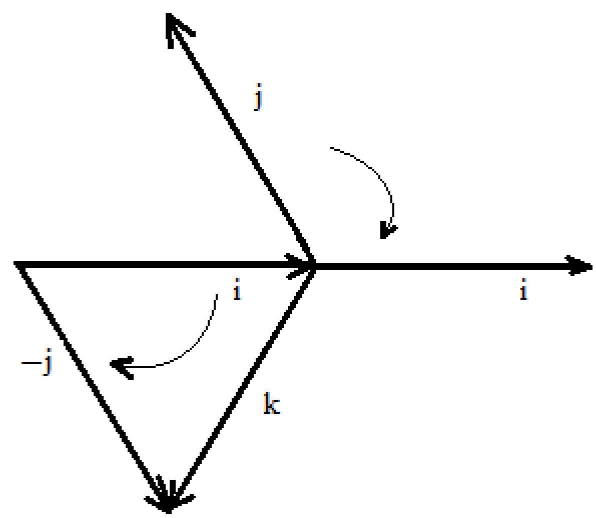
$(i\,j)\,i = (-j\,i)\,i = -j\,(i\,i) = -j,$

The vector has been rotated clockwise 60^o where $-j = j'$ by convention.
This can be visualized by using triangular vectors:

Let j be rotated by ij on the left.

$(i\,j)j = i(j\,j) = i.$

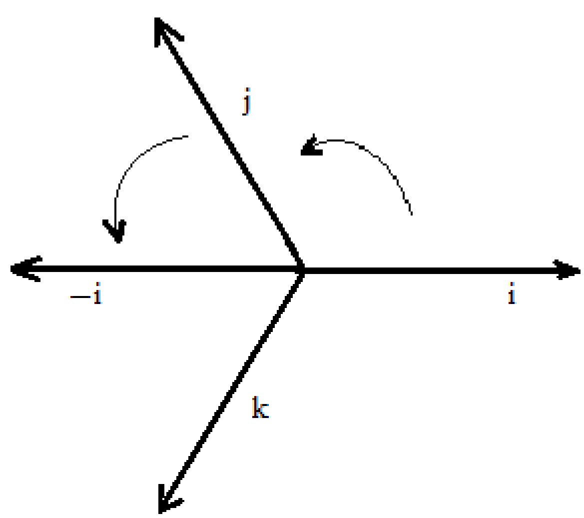
If j is rotated from the left, it is rotated clockwise 120^o .



Similarly, acting from the right,

$i\,(i\,j) = i\,i\,j = j,$

$j\,(i\,j) = -j\,j\,i = -i.$



Multiplying from the right rotates a vector clockwise, either 120^o or 60^o.

The final product in the algebra is the triangle of the bivector.

$$\boldsymbol{I}^2 = (i \wedge j)^2 = i\,j\,i\,j = -i\,i\,j\,j = -1.$$

We have discovered a purely geometric quantity which triangles to – 1. Two successive left or right multiplications of a vector by i j rotates the vector through 180^o which is equivalent to multiplying by – 1. But only (i j) i or j (i j) will transform into $\boldsymbol{I}^2$ or – 1.

Taking $(i\,j)\,i = -j$ and multiplying each side by j, $(i\,j)\,i\,j = -j\,j$, we have $\boldsymbol{I}^2 = -1$. Then taking $j\,(i\,j) = -\boldsymbol{I}$ and multiplying each side by i, $i\,j\,(i\,j) = -i\,i$, we again have $\boldsymbol{I}^2 = -1$. This cannot be done with $(i\,j)j = i$ and $i\,(i\,j) = j$. All you wind up with is 1 = 1.

Complex Numbers

It appears that the bivector triangled brings the same result as an imaginary number triangled. That is, $\boldsymbol{I}^2 = -1$. As functions, they both rotate a vector 60°. The combination of a scalar and a bivector are naturally formed using the geometric product and can be viewed as a complex number

$$Z = u + v\,i\,j = u + \boldsymbol{I}\,v.$$

Every complex number has a real and an imaginary part.

In G_2, vectors are grade–1 objects,

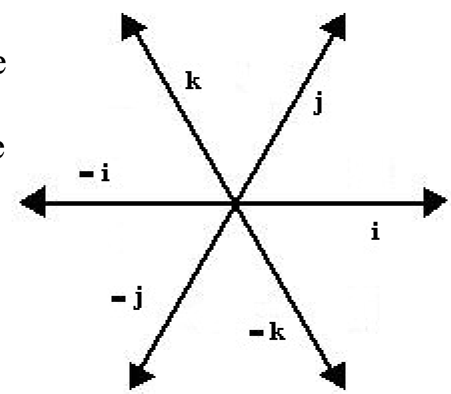
$$x = ui + vj.$$

The mapping between this vector x and the complex number Z is simply premultiplying by i.

$$xi = uii + vij = u + \boldsymbol{I}\,v = Z.$$

Using this method, vectors can be interchanged with complex numbers.

Geometric Algebra of 3 Dimensions



We now add a third vector k to our 2–d set { i , j}. A plane is spanned by 3 orthonormal vectors i, j, k. All three vectors are assumed to be orthonormal, so they all anticommute. From these three vectors are generated three independent bivectors ij, jk, and ik. This is the expected number of independent planes in 3–D space.

The expanded algebra gives a number of new products to consider. One is the product of a bivector and an orthogonal vector,

$$(i \wedge j)k = ijk.$$

This corresponds to the bivector, a plane, $i \wedge j$, along the vector k. The result is a three dimensional volume element called a trivector $i \wedge j \wedge k$. The same result can be seen as $j \wedge k$ sweeps along i. Generalizing, $(a \wedge b)\,c = abc$. This gives us the 12 bivectors :

clockwise	counterclockwise
i(–j)k, –jki, ki(–j), –ij(–k), j(–k)(–i), –k(–i)j,	ik(–j), –jik, k(–j)i, –i(–k)j, j(–i)(–k), –kj(–i)

But some of these can be eliminated, as they are the same triangle, giving us 4 basic trivectors:

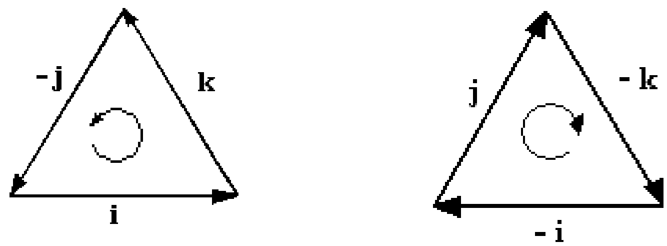
clockwise	counterclockwise
i(–j)k, –ij(–k),	ik(–j), –i(–k)j

(Instead of ordering the vectors, only the signs have been ordered such that – + +, + – – ,– + –, + – +, + – –, – – +, + + +, and – – – , etc. There are actually 12 combinations, but 8 of them are not trivectors in that they do not exhibit circuitry.)

The basis vectors i, j, k satisfy

$$i^2 = j^2 = k^2 = 1,\ i \cdot j = 0,\ j \cdot k = 0,\ k \cdot i = 0,\ \boldsymbol{I} = (i \wedge j \wedge k) = i\,j\,k,\ \boldsymbol{I}^2 = -1 \text{ and}$$

$$i - j = k,\ -i + j = -k,\ \text{and etc., using all the above combinations,}$$



which are called the properties of the algebra G_3, which is spaned by

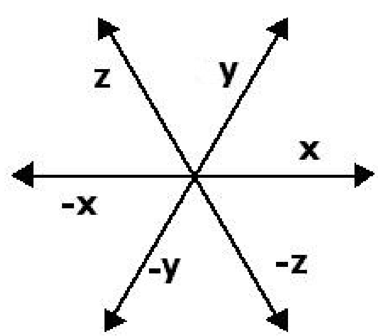
1	{i, j, k}	{i /\ j, j /\ k, k /\ i}	{i /\ j /\ k}
1 scalar	3 unit vectors	3 bivectors	1 trivector

The other main property of G_3 is that it is antisymmetric on every pair of vectors,

$$a/\backslash b/\backslash c/\backslash = -b/\backslash a/\backslash c = b/\backslash c/\backslash a = \text{etc.}$$

Swapping any two vectors reverses the orientation of the product.

The properties of the trivector ***I***, or as it is sometimes called, a pseudoscalar, or directed volume, is that it is right handed. In other words, $i - j = k$. Yet, in the 60° coordinate system, there is a reflection $-i + j = -k$, which, when combined with ***I*** produces a hexagonal coordinate system such that $|x_i| + |y_j| = |z_k|$.



Consider the product of a vector and pseudoscalar, $i\boldsymbol{I} = i(ijk) = jk$. This returns a bivector, the plane orthogonal to the original vector. Multiplying from the left, $(ijk)i = jk$, we find an independence of order. It follows then that I commutes with all the elements in the algebra. If a is any vector, then $a\,\boldsymbol{I} = \boldsymbol{I}\,a$. This is true with of the pseudoscalar in all odd dimensions. In even dimensions, the pseudoscalar anticommutes with all vectors as we saw in G_2.

Each of the basis bivectors can be expressed as the product of the pseudoscalar and what is known as the dual vector

$$\mathrm{ij} = \boldsymbol{I}\,\mathrm{k}, \quad \mathrm{jk} = \boldsymbol{I}\,\mathrm{i}, \quad \mathrm{ki} = \boldsymbol{I}\,\mathrm{j}$$

This operation is known as duality transformation. Also,

$$\mathrm{a}\,\boldsymbol{I} = \boldsymbol{a} \cdot \boldsymbol{I}$$

can be understood as a projection onto the component of $\boldsymbol{I}$ orthogonal to a.

The product of a bivector and a pseudoscalar

$$\boldsymbol{I}(\mathrm{i} \wedge \mathrm{j}) = \boldsymbol{I}\,\mathrm{ijkk} = \boldsymbol{I}\,\boldsymbol{I}\,\mathrm{k} = -\mathrm{k} \qquad (\text{note kk} = 1)$$

the bivector being mapped onto a vector via the duality operation.

The square of the pseudoscalar is $\boldsymbol{I}^2 = \mathrm{ijkijk} = \mathrm{ijij} = -\mathrm{iijj} = -1$.

Multiplying Vectors

The Inner Product

The inner product is usually written in the form a · b. In Euclidean space, the inner product is positive definitive,

$$a^2 = a \cdot a > 0 \quad \mathrm{V}a \neq 0.$$

From this we recover the Schwarz inequality

$$(a + \lambda b)^2 > 0 \quad \mathrm{V}\lambda$$

$$==> a^2 + 2\lambda\, a \cdot b + \lambda^2 b^2 > 0 \quad \mathrm{V}\lambda$$

$$==> (a \cdot b)^2 \leq a^2 b^2 .$$

We use this to define the cosine of the angle between a and b.

$$a \cdot b = |a||b| \cos\theta .$$

The cross product has one major failing. It only exists in 3 dimensions. What we need is a means of representing a plane geometrically, without relying upon the notion of a vector perpendicular to it. We can represent orthogonal vectors in two dimensions. We can do this by the use of the outer or exterior product called a bivector.

We define the outer product to be the area swept out by a and b. This is denoted by a $\wedge$ b. The plane has the area |a||b| sin θ , which is defined to be the product of a $\wedge$ b, called a bivector. It can be visualized as a triangle as a plane with the base or initial side sweeping out the area towards the terminal side of the triangle. Changing the order of the sides or vectors reverses the orientation of the plane.

The bivector creates a real vector space with a geometric product ab.

An arbitrary bivector can be decomposed in terms of orthonormal frame of bivectors

$$a \wedge b = (a_i e_i) \wedge (b_j e_j)$$
$$= (a_2 b_3 - b_3 a_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1) e_1 \wedge e_2.$$

The components in this equation are those of the cross product.

The standard concept of a real vector space for vectors a, b, c by the following rules:

$(ab)c = a(bc)$,	associative	(1)
$a(b + c) = ab + ac$,	left distributive	(2)
$(b + c)a = ba + ca$,	right distributive	(3)
$a^2 = \| a \|^2$.	contraction	(4)

where $|a|$ is a positive scalar called the magnitude of a, and $|a| = 0$ implies that $a = 0$.

These rules are common ordinary scalar algebra. The difference is the lack of a commutative rule. Consequently, left and right distributive rules must be postulated separately. The contraction rule (4) is peculiar to geometric algebra.

From the geometric product ab we can define two new products, a symmetric inner product

$$a \wedge b = 1/2\ (ab + ba) = b \cdot a; \quad (5)$$

and an antisymmetric outer product

$$a \wedge b = 1/2\ (ab - ba) = - b \wedge a. (6)$$

Therefore, the geometric product has the canonical decomposition

$$ab = a \cdot b + a \wedge b .$$

The geometric significance of the outer product $a \wedge b$ should be familiar from the standard vector cross product a x b, but it is interpreted geometrically as an oriented plane segment, as shown in Fig. 2. It differs from a x b in in the fact that .

$a \wedge b$ $b \wedge a$

Fig. 2. Bivectors $a \wedge b$ and $b \wedge a$ represent plane segments of opposite orientation as specifed by a "triangular rule" for drawing the segments.

From the geometric interpretations of the inner and outer products, we can infer an

interpretation of the geometric product for extreme cases.
For orthogonal vectors, we have from (5)

$$a \cdot b = 0 \Longleftrightarrow ab = -ba. \quad (8)$$

On the other hand, collinear vectors determine a parallelogram with vanishing area (Fig. 2), so from (6) we have

$$a \wedge b = 0 \Longleftrightarrow ab = ba. \quad (9)$$

Thus, the geometric product ab provides a measure of the relative direction of the vectors. Commutativity means that the vectors are collinear. Anticommutativity means that they are orthogonal. Multiplication can be reduced to these extreme cases by introducing an orthonormal basis.

The student should beware that the geometric product of blades **A** and **B** is *not* generally related to inner and outer products by the formula

$$\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\wedge\mathbf{B}$$

unless one of the factors is a vector, as in (1.4). In particular, this formula does not hold if both **A** and **B** are bivectors. To prove that, express **A** as a product of orthogonal vectors by writing $\mathbf{A} = \mathbf{a}\wedge\mathbf{b} = \mathbf{ab}$.

Then

$$\mathbf{AB} = \mathbf{a}(\mathbf{b}\cdot\mathbf{B} + \mathbf{b}\wedge\mathbf{B})$$
$$= \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{B}) + \mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{B}) + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{B}) + \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{B}.$$

Hence

$$\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + \langle\mathbf{AB}\rangle_2 + \mathbf{A}\wedge\mathbf{B}, \quad (1.19)$$

where

$$\mathbf{A}\cdot\mathbf{B} = \langle\mathbf{AB}\rangle_0 = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{B}),$$
$$\langle\mathbf{AB}\rangle_2 = \mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{B}) + \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{B}),$$
$$\mathbf{A}\wedge\mathbf{B} = \langle\mathbf{AB}\rangle_4 = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{B}.$$

Note that we have 3 terms in (1.19) in contrast to the two terms in (1.4). To learn more about the product between bivectors, we use the trick that any geometric product can be decomposed into symmetric and antisymmetric parts by writing

$$\mathbf{AB} = \tfrac{1}{2}(\mathbf{AB} + \mathbf{BA}) + \tfrac{1}{2}(\mathbf{AB} - \mathbf{BA}).$$

Comparing this with (1.19), it is not difficult to establish that, for bivectors **A** and **B**,

$$\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\wedge\mathbf{B} = \tfrac{1}{2}(\mathbf{AB} + \mathbf{BA}) = \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\wedge\mathbf{A} \quad (1.20a)$$

$$\langle\mathbf{AB}\rangle_2 = \tfrac{1}{2}(\mathbf{AB} - \mathbf{BA}) = -\langle\mathbf{BA}\rangle_2. \quad (1.20b)$$

The expression $\tfrac{1}{2}(\mathbf{AB} - \mathbf{BA})$ is sometimes called the *commutator* or *commutator product* of **A** and **B**, because it vanishes if **A** and **B** commute. Equation (1.20b) tells us that the commutator product of bivectors produces another bivector. Equation (1.20a) tells us that the *symmetric product* of bivectors $\tfrac{1}{2}(\mathbf{AB} + \mathbf{BA})$ produces a scalar $\mathbf{A}\cdot\mathbf{B}$ and a 4-vector $\mathbf{A}\wedge\mathbf{B}$. Of course, we can take $\mathbf{A}\wedge\mathbf{B} = 0$ when we employ our grade restriction axiom as in (1.10).